Rational Functions and Asymptotes

A rational function looks like

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Ex. State the domain of $f(x) = \dfrac{x+3}{x-2}$.

$x \neq 2$

Look at the graph of the last example.

Note that the y-coordinate goes to infinity as x gets close to 2.

We say that $f(x)$ has a <u>vertical asymptote</u> of $x = 2$.

Note that the height approaches $y = 1$ as the graph goes left and right

We say that $f(x)$ has a <u>horizontal asymptote</u> of $y = 1$.

Vertical Asymptotes

- values of x that make the bottom equal 0…
- unless they cause the top to be zero as well

Horizontal Asymptotes

- look at lead terms on top and bottom
- ask what happens as $x \rightarrow \infty$

Ex. Find all asymptotes of

a) $f(x) = \dfrac{3x + 2x^2}{x^2 - 1} = \dfrac{x(3+2x)}{(x+1)(x-1)}$

h.a. as $x \to \infty$, $f \to \dfrac{2x^2}{x^2} = 2$

$\boxed{y = 2}$

v.a. $(x+1)(x-1) = 0$

$\boxed{x = -1}$ $\boxed{x = 1}$

b) $f(x) = \dfrac{x^2 + x - 2}{x^2 - x - 6} = \dfrac{\cancel{(x+2)}(x-1)}{(x-3)\cancel{(x+2)}} = \dfrac{x-1}{x-3}$

$x \neq 3, -2$

h.a. as $x \to \infty$, $f \to \dfrac{x^2}{x^2} = 1$

$\boxed{y = 1}$

v.a. $(x-3)(x+2) = 0$

$\boxed{x = 3}$ $\cancel{x = -2}$

$f(x) = \dfrac{x-1}{x-3} \Longrightarrow f(-2) = \dfrac{-2-1}{-2-3} = \dfrac{3}{5}$

Graphing Rational Functions

1) Evaluate $f(0)$ – Gives y-intercept
2) Factor top and bottom – cancelled factors are holes
3) Zeroes of the top – Gives x-intercepts
4) Zeroes of the bottom – Gives vert. asympt.
5) Let $x \rightarrow \infty$ – Gives horiz. asympt.
6) Plot more points if needed

$\underline{\text{Ex.}}$ Graph $f(x) = \dfrac{3}{x-2}$

1) $f(0) = \dfrac{3}{0-2} = -\dfrac{3}{2}$ ← y-int.

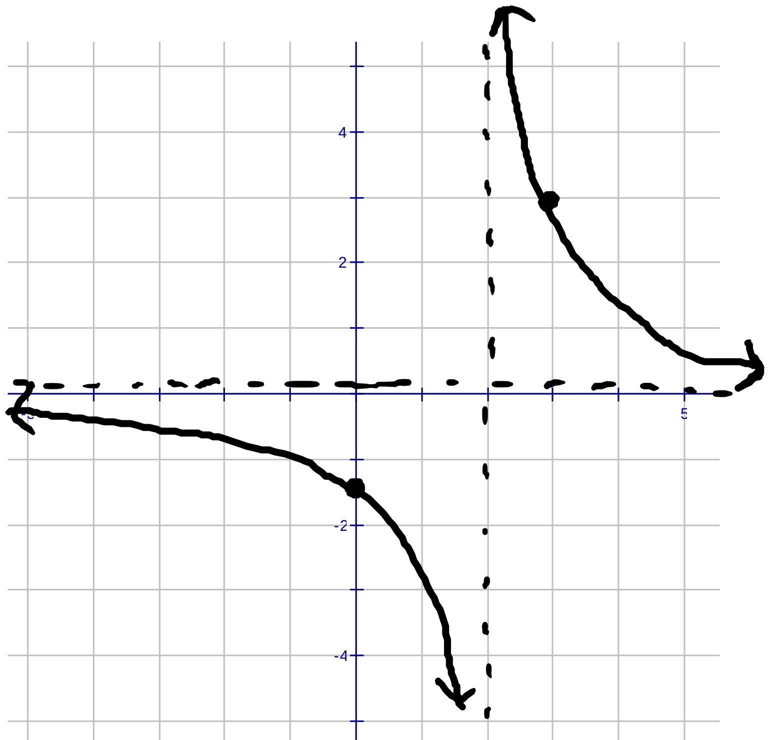
2) ✓

3) $\cancel{3 = 0}$ ← no x-int.

4) $x - 2 = 0$
$x = 2$ ← vert. asymp.

5) as $x \to \infty$, $f \to \dfrac{3}{x} = 0$
$y = 0$ ← horiz. asymp.

6) $f(3) = \dfrac{3}{3-2} = 3$



Ex. Graph $f(x) = \dfrac{2x-1}{x}$

1) $f(0) = \dfrac{2(0)-1}{0} = \times$ ← no y-int.

2) ✓

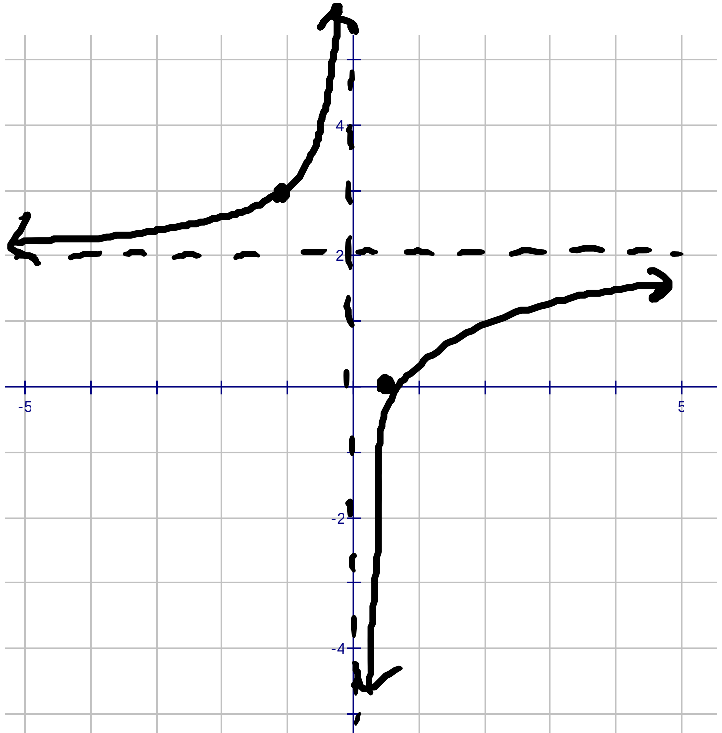
3) $2x-1=0 \rightarrow x=\frac{1}{2}$ ← x-int.

4) $x=0$ ← vert. asymp.

5) as $x \to \infty$, $f \to \dfrac{2x}{x} = 2$

$y=2$ ← horiz. asymp.

6) $f(-1) = \dfrac{2(-1)-1}{-1} = \dfrac{-3}{-1} = 3$



Ex. Graph $f(x)=\dfrac{x}{x^2-x-2} = \dfrac{x}{(x-2)(x+1)}$

1) $f(0)=\dfrac{0}{0^2-0-2}=\dfrac{0}{-2}=0 \leftarrow$ y-int.

2) ✓

3) $x=0 \leftarrow$ x-int.

vert. asymp.

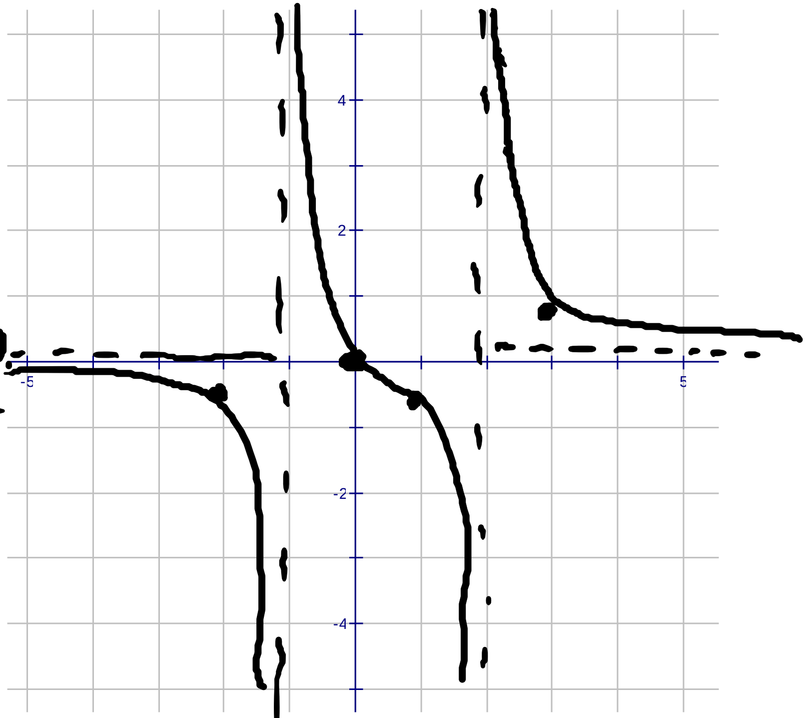
4) $(x-2)(x+1)=0 \Rightarrow x=2,\ x=-1$

5) as $x\to\infty$, $f\to \dfrac{x}{x^2}=\dfrac{1}{x}=0$

$y=0 \leftarrow$ horiz. asymp.

6) $f(3)=\dfrac{3}{4}$ $\quad f(1)=\dfrac{1}{-2}$

$f(-2)=\dfrac{-2}{4}=-\dfrac{1}{2}$



$\underline{\text{Ex.}}$ Graph $f(x) = \dfrac{x^2 - 9}{x^2 - 2x - 3} = \dfrac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+1)} = \dfrac{x+3}{x+1}$

$\boxed{x = 3 \leftarrow \text{hole}}$

1) $f(0) = \dfrac{0^2 - 9}{0^2 - 2(0) - 3} = \dfrac{-9}{-3} = 3 \leftarrow$ y-int.

2) ✓

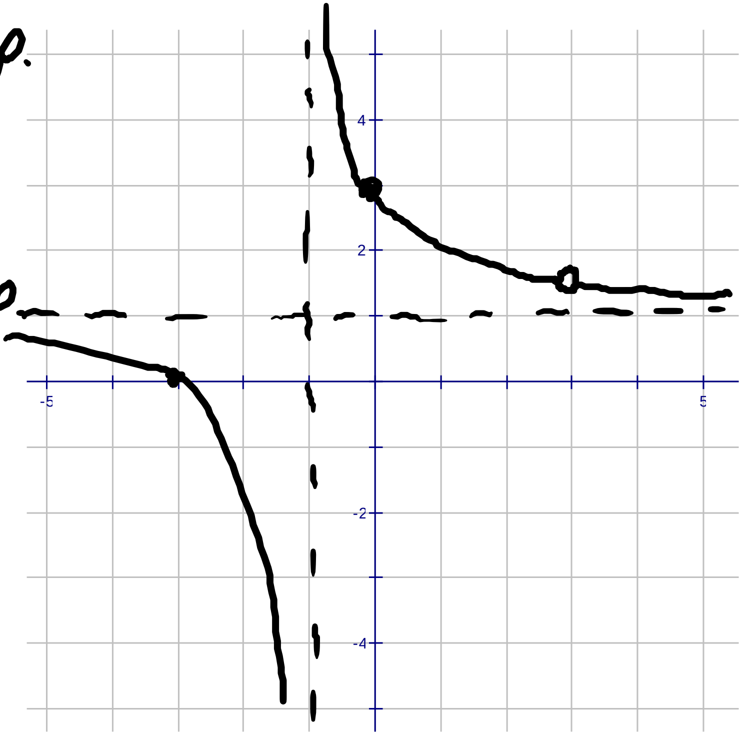
3) $x + 3 = 0 \Rightarrow x = -3 \leftarrow$ x-int.

4) $x + 1 = 0 \Rightarrow x = -1 \leftarrow$ vert. asymp.

5) as $x \to \infty$, $f \to \dfrac{x}{x} = 1$

$y = 1 \leftarrow$ horiz. asymp.

6) $f(3) = \dfrac{3+3}{3+1} = \dfrac{6}{4} = \dfrac{3}{2}$



When the degree on top is one more than the degree on the bottom, there won't be a horizontal asymptote.

However, if we do the division, we can find a <u>slant asymptote</u>.

$\underline{\text{Ex.}}$ Graph $f(x) = \dfrac{x^2 - x - 2}{x - 1} = \dfrac{(x-2)(x+1)}{x-1}$

1) $f(0) = \dfrac{0-0-2}{0-1} = 2$ ← y-int.

2) ✓

3) $(x-2)(x+1) = 0 \Rightarrow x = 2,\ x = -1$ ← x-int.

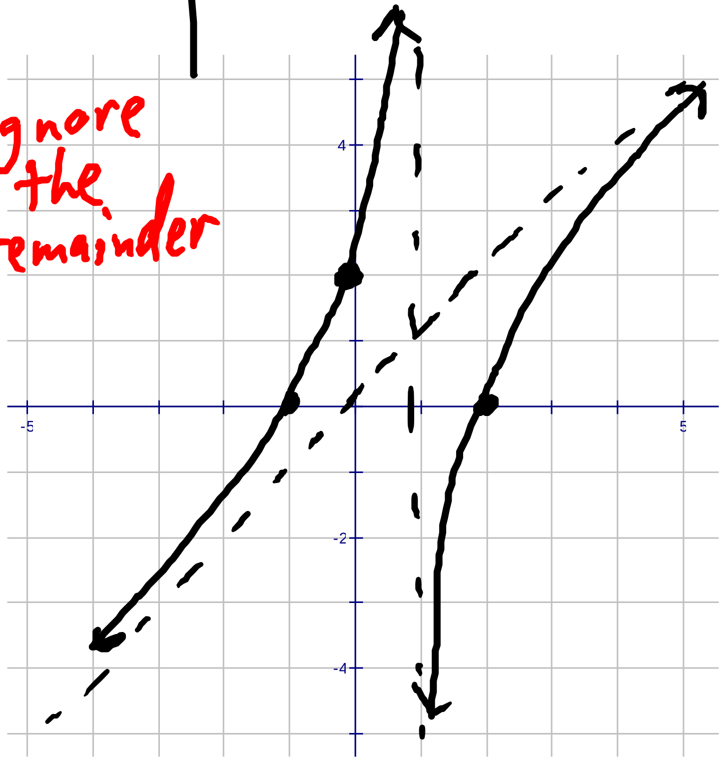
4) $x - 1 = 0 \rightarrow x = 1$ ← vert. asymp.

5)
$$\begin{array}{c|ccc} 1 & 1 & -1 & -2 \\ & & 1 & 0 \\ \hline & 1 & 0 & \boxed{-2} \end{array}$$

$f(x) = x + \cancel{\dfrac{-2}{x-1}}$ ignore the remainder

$y = x$ ← slant asymp.

$$x-1\,\overline{)\,x^2 - x - 2}$$



<u>Ex.</u> A rectangular page with margins shown below is designed to have 48 in^2 of print. What should the dimensions be for the page that uses the least paper?