First part is out of 25 pts.

Second part is out of 80 pts.

Total points possible is 105 pts.

→ Grade is out of 100 pts.

Exponential Functions

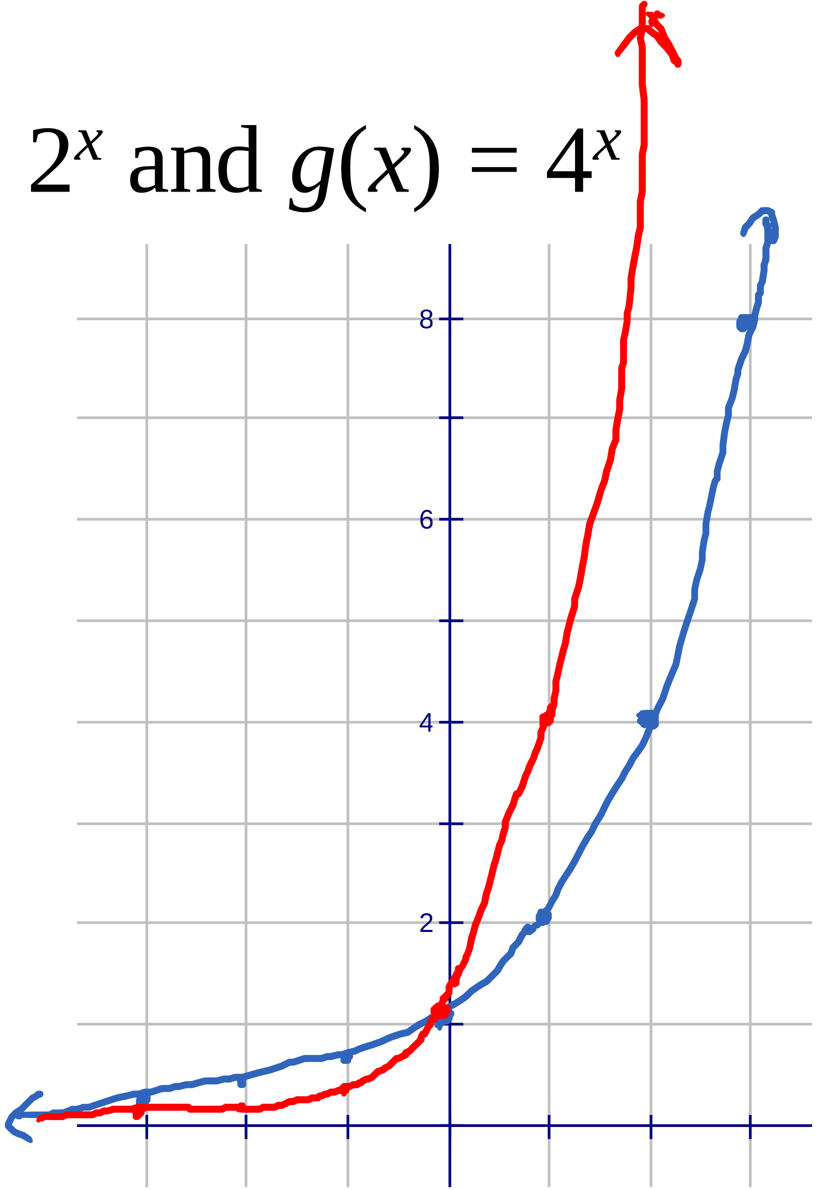
An exponential function is of the form $f(x) = a^x$, where $a > 0$. a is called the base.

Ex. Let $h(x) = 3.1^x$, evaluate $h(-1.8)$.

$h(-1.8) = 3.1^{-1.8} = .13$

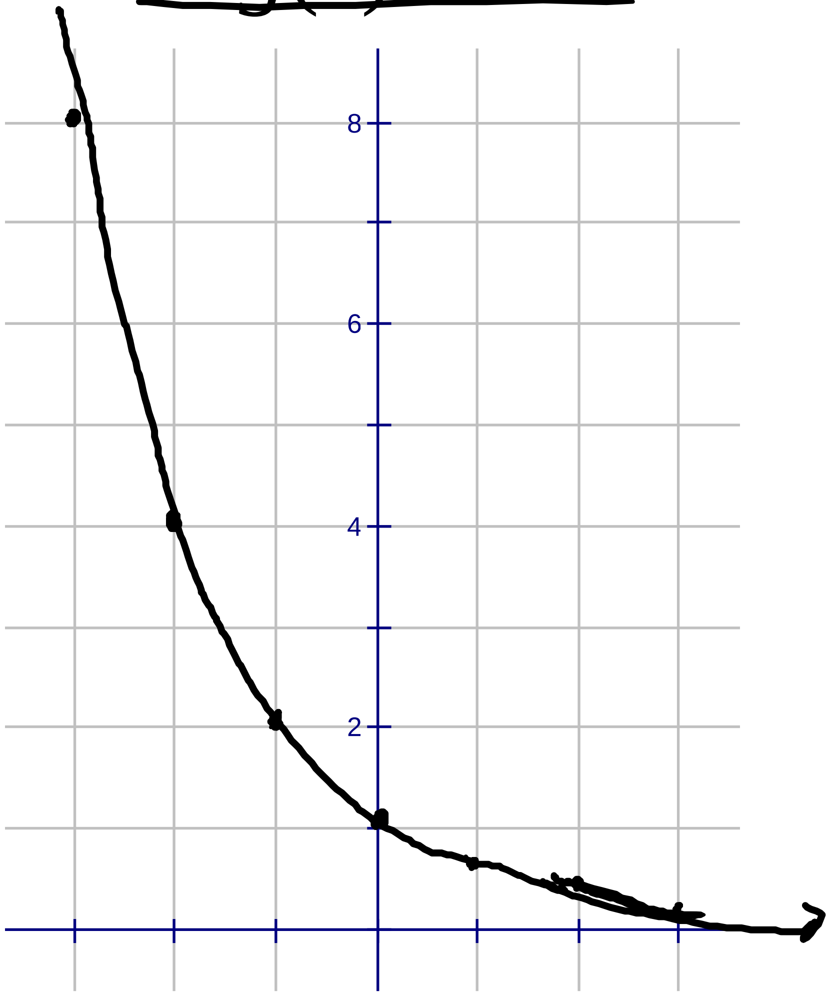
Ex. Sketch $f(x) = 2^x$ and $g(x) = 4^x$

x	$f(x) = 2^x$	$g(x) = 4^x$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$4^{-3} = \frac{1}{4^3} = \frac{1}{64}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$4^{-2} = \frac{1}{4^2} = \frac{2}{16}$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$4^{-1} = \frac{1}{4^1} = \frac{1}{4}$
0	$2^0 = 1$	$4^0 = 1$
1	$2^1 = 2$	$4^1 = 4$
2	$2^2 = 4$	$4^2 = 16$
3	$2^3 = 8$	$4^3 = 64$



Ex. Sketch $f(x) = 2^{-x}$ ~~and $g(x) = 4^{-x}$~~

x	$f(x) = 2^{-x}$
-3	$2^{-(-3)} = 2^3 = 8$
-2	$2^{-(-2)} = 2^2 = 4$
-1	$2^{-(-1)} = 2^1 = 2$
0	$2^{-0} = 1$
1	$2^{-1} = \frac{1}{2}$
2	$2^{-2} = \frac{1}{4}$
3	$2^{-3} = \frac{1}{8}$



Note that every exponential function has

- Domain of all reals
- Range of $y > 0$
- Horizontal asymptote at $y = 0$

Because they are monotonic, these functions are one-to-one

Let $a > 0$ and $a \neq 1$. If $a^x = a^y$, then $x = y$.

Ex. Solve for x.

a) $9^x = 3^{x+1}$

$(3^2)^x = 3^{x+1}$

$3^{2x} = 3^{x+1}$

$2x = x + 1$

$\boxed{x = 1}$

b) $(\frac{1}{2})^x = 8$

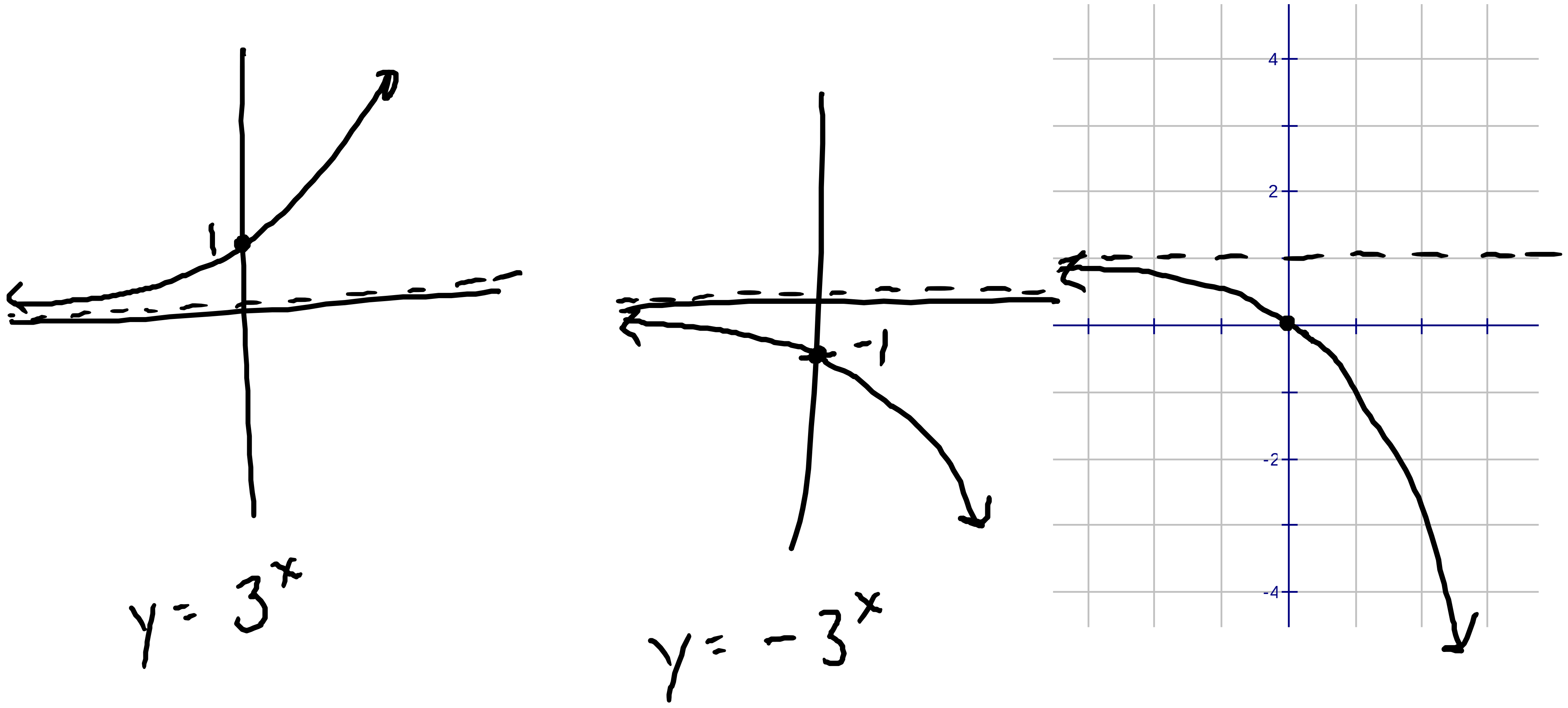
$(2^{-1})^x = 2^3$

$2^{-x} = 2^3$

$-x = 3$

$\boxed{x = -3}$

Ex. Sketch the graph of $f(x) = 1 - 3^x$. $= -3^x + 1$



The number $e \approx 2.718281828$ is called the natural base and the function $f(x) = e^x$ is called the natural exponential function.

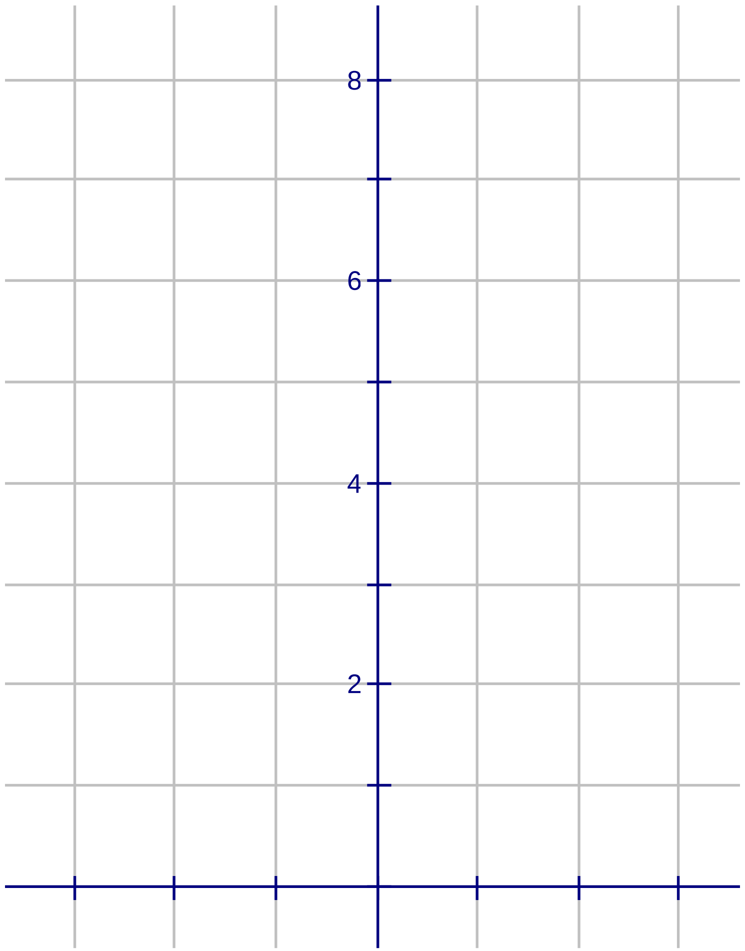
Euler

We will see this function come up in some application problems, especially investment problems where interest is compounded continuously.

Ex. Let $f(x) = e^x$, evaluate $f(3.2)$

$f(3.2) = e^{3.2} = 24.53$

Ex. Use a calculator to graph $f(x) = 2e^{.24x}$ and $g(x) = \frac{1}{2}e^{-.58x}$



When interest is compounded n times per year, we used the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$A =$ amount in bank

$P =$ principal invested

$r =$ interest rate

$t =$ time (in years)

If interest is compounded continually, we use the formula

$$A = Pe^{rt}$$

Ex. You invest $12,000 at an annual rate of 9%. How much more can you make in 5 years if interest is compounded continually rather than monthly?

$= P$ $\quad r = .09 \quad t = 5$

monthly

$n = 12$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$= 12000\left(1 + \frac{.09}{12}\right)^{12 \cdot 5}$$

$$= 18{,}788.17$$

continuously

$$A = Pe^{rt}$$

$$= 12000e^{(.09)(5)}$$

$$= 18{,}819.75$$

difference = \$31.58

Ex. After the Chernobyl nuclear accident in 1986, the amount remaining after t years from a 10 pound sample of plutonium can be modeled by the function

$$P = 10\left(\tfrac{1}{2}\right)^{t/24,100}$$

How much of this sample remains today?

$t = 35$

$$P = 10\left(\tfrac{1}{2}\right)^{35/24100} = 9.9899$$

Logarithmic Functions

We saw that exponential functions are invertible.

The logarithmic function $f(x) = \log_a x$ is the inverse function of $g(x) = a^x$.

$$y = \log_a x \quad \leftrightarrow \quad a^y = x$$

Ex. Evaluate by hand.

a) $\cancel{\log}_2 32 = x \longrightarrow 32 = 2^x \longrightarrow x = 5$

b) $\cancel{\log}_3 1 = x \longrightarrow 1 = 3^x \longrightarrow x = 0$

c) $\cancel{\log}_9 3 = x \longrightarrow 3 = 9^x \longrightarrow 3 = (3^2)^x$

$3^1 = 3^{2x} \longrightarrow 1 = 2x$

$\boxed{x = \frac{1}{2}}$

d) $\cancel{\log}_{10} \frac{1}{100} = x \longrightarrow \frac{1}{100} = 10^x \longrightarrow \frac{1}{10^2} = 10^x$

$10^{-2} = 10^x \longrightarrow \boxed{x = -2}$

The logarithmic function with base 10 is called the common logarithm is can be written $f(x) = \log x$

Ex. Given the function $f(x) = \log x$, evaluate $f(2.5)$ and $f(-2)$.

$f(2.5) = \log(2.5) = 0.398$

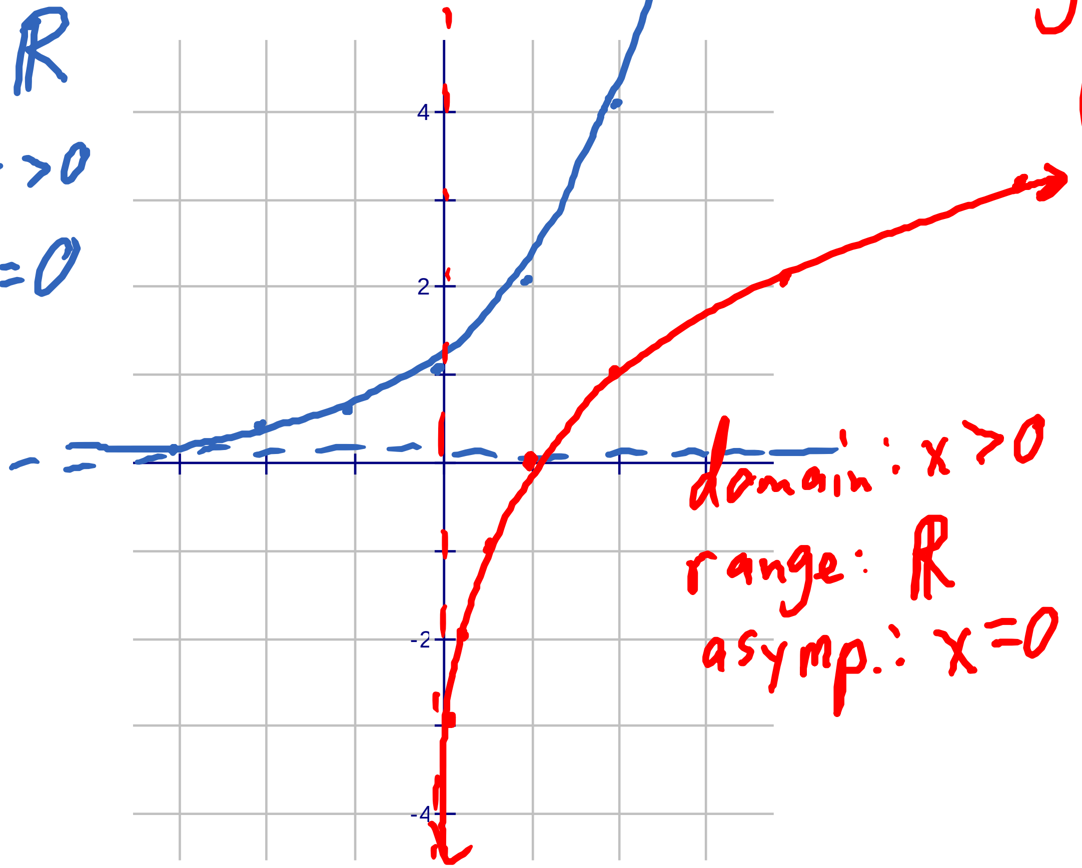
$f(-2) = \log(-2) = ???$

Ex. Use the graph of $f(x) = 2^x$ to sketch $g(x) = \log_2 x$.

$f(x) = 2^x$

$(-3, \frac{1}{8})$
$(-2, \frac{1}{4})$
$(-1, \frac{1}{2})$
$(0, 1)$
$(1, 2)$
$(2, 4)$
$(3, 8)$

domain: $\mathbb{R}$
range: $y > 0$
asymp.: $y = 0$



domain: $x > 0$
range: $\mathbb{R}$
asymp.: $x = 0$

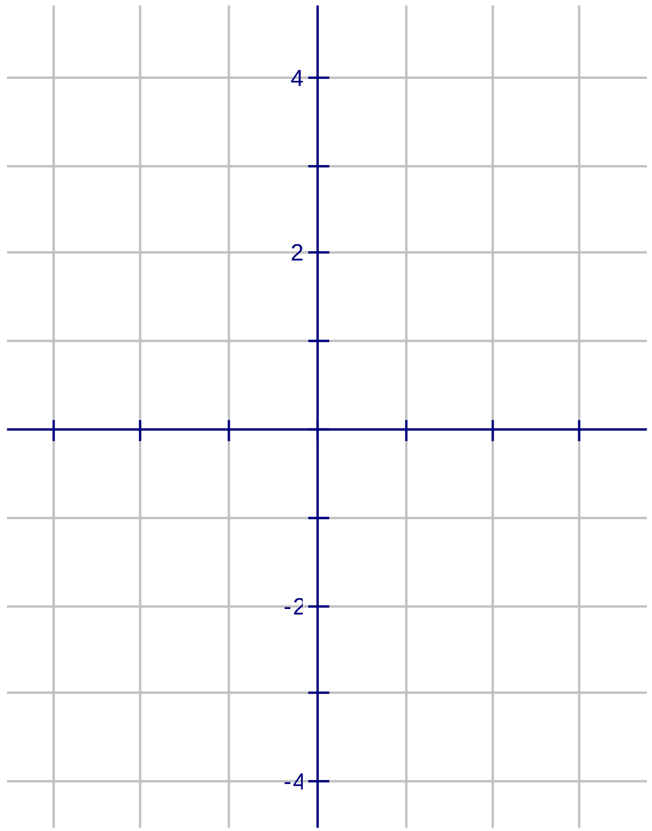
$g(x) = \log_2 x$

$(\frac{1}{8}, -3)$
$(\frac{1}{4}, -2)$
$(\frac{1}{2}, -1)$
$(1, 0)$
$(2, 1)$
$(4, 2)$
$(8, 3)$

Note that every logarithmic function has

- Domain of $x > 0$
- Range of all reals
- Vertical asymptote at $x = 0$

Ex. Sketch the graph of $f(x) = 1 - \log x$



Properties of Logarithms

$a^0 = 1 \quad \leftrightarrow \quad \log_a 1 = 0$

$a^1 = a \quad \leftrightarrow \quad \log_a a = 1$

Since exponents and logarithms are inverse, $a^{\log_a x} = x$ and $\log_a a^x = x$

Since logarithms are one-to-one, we know:

If $\log_a x = \log_a y$, then $x = y$.

Ex. Simplify

a) $\log_4 1 = 0$

b) $\log_8 8 = 1$

c) $\log_6 6^{20} = 20$

$\underline{\text{Ex.}}$ Solve

a) $\log_3 4x = \log_3 20$

$4x = 20$

$x = 5$

b) $\log (2x + 1) = \log x$

$2\cancel{x} + 1 = x$

$-\cancel{2}x \quad -2x$

$1 = -x$

$\cancel{x = -1}$

no solution

The function $f(x) = \log_e x$ is called the <u>natural logarithm function</u>, and it is often written

$$f(x) = \ln x$$

<u>Ex.</u> Evaluate the natural log function at $x = 3$ and $x = -4$

$f(3) = \ln 3 = 1.10$

$f(-4) = \ln(-4) = ??$

Ex. Simplify

a) $\ln \dfrac{1}{e} = \ln_e(e^{-1}) = -1$

b) $\ln e^5 = 5$

c) $8\underbrace{\ln 1}_{0} = 0$

d) $2\underbrace{\ln e^1}_{1} = 2$

Ex. Identify the domain

a) $f(x) = \ln(x - 2)$

$x - 2 > 0$

$\boxed{x > 2}$

b) $f(x) = \ln(2 - x)$

$2 - x > 0$

$2 > x$

$\boxed{x < 2}$

c) $f(x) = \ln(x^2)$

$x \neq 0$