Partial Fractions

We are going to write rational functions as the sum of fractions with smaller denominators

$$\frac{x+7}{x^2-x-6}=\frac{2}{x-3}+\frac{-1}{x+2}$$

This is called partial fraction decomposition

Factor the bottom (not the top) completely, these factors become the denominators of the new fractions

For this to work, we need the top to have a smaller degree than the bottom

- If the top has the same or larger degree, use long division
- Then, use partial fraction decomposition on the remainder

<u>Ex.</u> Decompose $\dfrac{x+7}{x^2-x-6}$

$(x-3)(x+2)$

$$\frac{x+7}{(x-3)(x+2)} = \frac{A(x+2)}{(x-3)(x+2)} + \frac{B(x-3)}{(x+2)(x-3)}$$

$$\frac{x+7}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$x+7 = A(x+2)+B(x-3)$$

$$x+7 = Ax+2A+Bx-3B$$

$$1x+7 = (A+B)x+(2A-3B)$$

$$A+B=1$$

$$2A-3B=7$$

Ex. Decompose $\dfrac{x+7}{x^2-x-6}$ ~~as a~~ $\rightarrow$ $\boxed{\dfrac{2}{x-3}+\dfrac{-1}{x+2}}$

$(x-3)(x+2)$

$$\frac{x+7}{(x-3)(x+2)} = \frac{A(x+2)}{(x-3)(x+2)} + \frac{B(x-3)}{(x+2)(x-3)}$$

$$\frac{x+7}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$x+7 = A(x+2) + B(x-3)$$

$\underline{x=3:}$ $\quad 10 = A(5) \longrightarrow A=2$

$\underline{x=-2:}$ $\quad 5 = B(-5) \longrightarrow B=-1$

$\underline{\text{Ex.}}$ Decompose $\dfrac{5}{x^2-7x+12} = \dfrac{-5}{x-3} + \dfrac{5}{x-4}$

$(x-3)(x-4)$

$$\frac{5}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

$$\frac{5}{(x-3)(x-4)} = \frac{A(x-4)+B(x-3)}{(x-3)(x-4)}$$

$$5 = A(x-4)+B(x-3)$$

$\underline{x=3:}$ $5 = A(-1) \rightarrow A = -5$

$\underline{x=4:}$ $5 = B(1) \rightarrow B = 5$

<u>Ex.</u> Decompose $\boxed{\dfrac{x^4+2x^3+6x^2+20x+6}{x^3+2x^2+x}}$

$$\begin{array}{r} x \\ x^3+2x^2+x \overline{\smash{)}\, x^4+2x^3+6x^2+20x+6} \\ {}^-x^4+{}^-2x^3+{}^-x^2 \\ \hline 5x^2+20x+6 \end{array}$$

$$= x + \frac{5x^2+20x+6}{x^3+2x^2+x}$$

$$= x + \frac{5x^2+20x+6}{x^3+2x^2+x} = \boxed{x + \frac{6}{x} + \frac{-1}{x+1} + \frac{9}{(x+1)^2}}$$

$$x(x^2+2x+1)$$

$$x(x+1)^2$$

$$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A(x+1)^2}{x(x+1)^2} + \frac{B\,x(x+1)}{(x+1)\,x(x+1)} + \frac{C\,x}{(x+1)^2\,x}$$

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

$\underline{x=-1:}$ $\quad 5-20+6 = C(-1) \longrightarrow C=9$

$\underline{x=0:}$ $\quad 6 = A(1) \longrightarrow A=6$

$\underline{x=1:}$ $\quad 5+20+6 = 6(2)^2 + B(1)(2) + 9(1)$

$$31 = 24 + 2B + 9$$

$$31 = 2B + 33$$

$$2B = -2 \longrightarrow B = -1$$

$$\text{Degree } 2 \rightarrow \quad \frac{\qquad}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\text{Degree } 4 \rightarrow \quad \frac{\qquad}{(x+1)(x+2)^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

$$\text{Degree } 3 \rightarrow \quad \frac{\qquad}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$$

$$\text{Degree } 5 \rightarrow \quad \frac{\qquad}{(x+1)(x^2+2)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

Ex. Decompose $\dfrac{10x^2+3x+3}{(x^2+4)(x+1)} = \dfrac{8x-5}{x^2+4} + \dfrac{2}{x+1}$

$$\frac{10x^2+3x+3}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}$$

$$10x^2+3x+3 = (Ax+B)(x+1) + C(x^2+4)$$

$\underline{x=-1:}$ $10-3+3 = C(5) \longrightarrow C=2$

$\underline{x=0:}$ $0+0+3 = B(1)+2(4) \longrightarrow B=-5$

$\underline{x=1:}$ $10+3+3 = (A-5)(2)+2(5)$

$$16 = 2A-10+10$$

$$16 = 2A$$

$$A = 8$$

$\underline{\text{Ex.}}$ Decompose $\dfrac{8x^3+13x}{(x^2+2)^2} = \dfrac{8x}{x^2+2} + \dfrac{-3x}{(x^2+2)^2}$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{(Ax+B)(x^2+2)}{(x^2+2)(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}$$

$$8x^3+13x = (Ax+B)(x^2+2) + (Cx+D)$$

$$8x^3+13x = Ax^3+Bx^2+2Ax+2B+Cx+D$$

$$8x^3+13x = Ax^3+Bx^2+(2A+C)x+(2B+D)$$

$A=8$ $\quad B=0 \quad$ $2A+C=13$ $\quad$ $2B+D=0$

$2(8)+C=13$ $\quad$ $2(0)+D=0$

$C=-3$ $\quad$ $D=0$