Systems of Inequalities

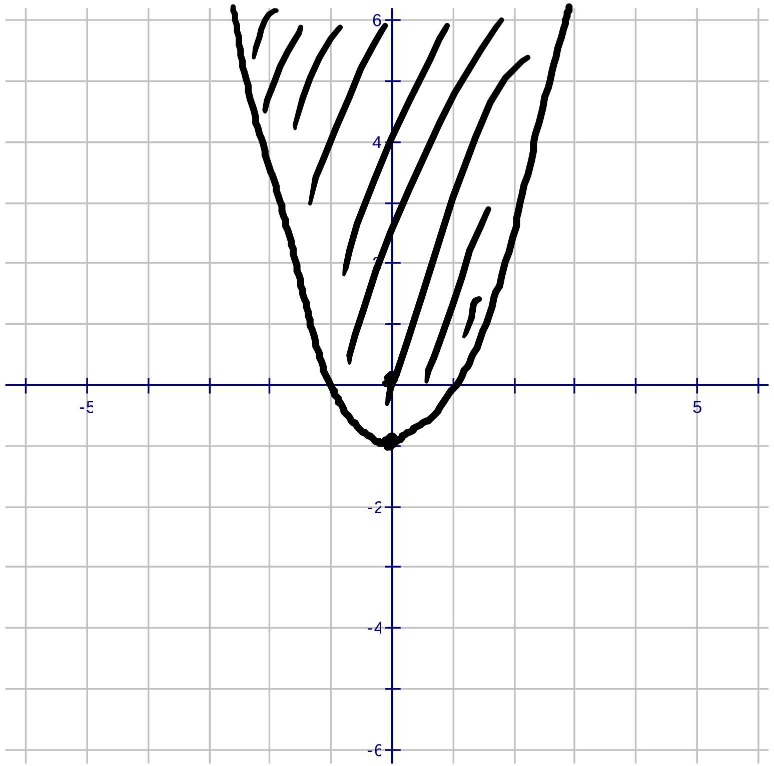
We are going to sketch inequalities ($<$, $>$, $\leq$, $\geq$) the same way we did before...with a test point

Ex. Sketch $y \geq x^2 - 1$

$y = x^2 - 1$

Test: $(0, 0)$

$0 \geq 0^2 - 1$ true



Ex. Sketch the graph of $x - y < 2$

$x - y = 2$

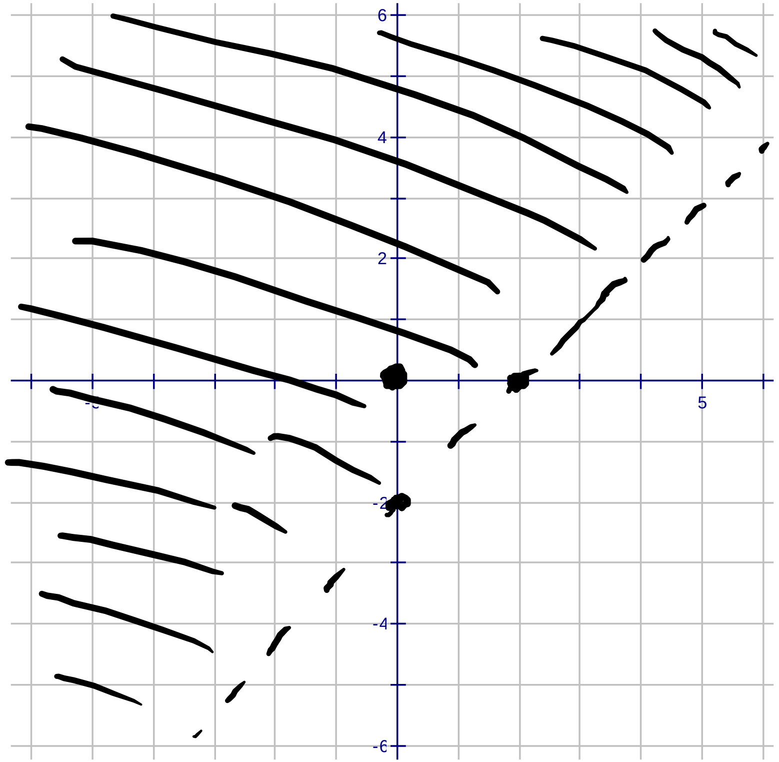
$x = 0: \quad -y = 2 \Rightarrow y = -2 \qquad (0, -2)$

$y = 0: \quad x = 2 \qquad (2, 0)$

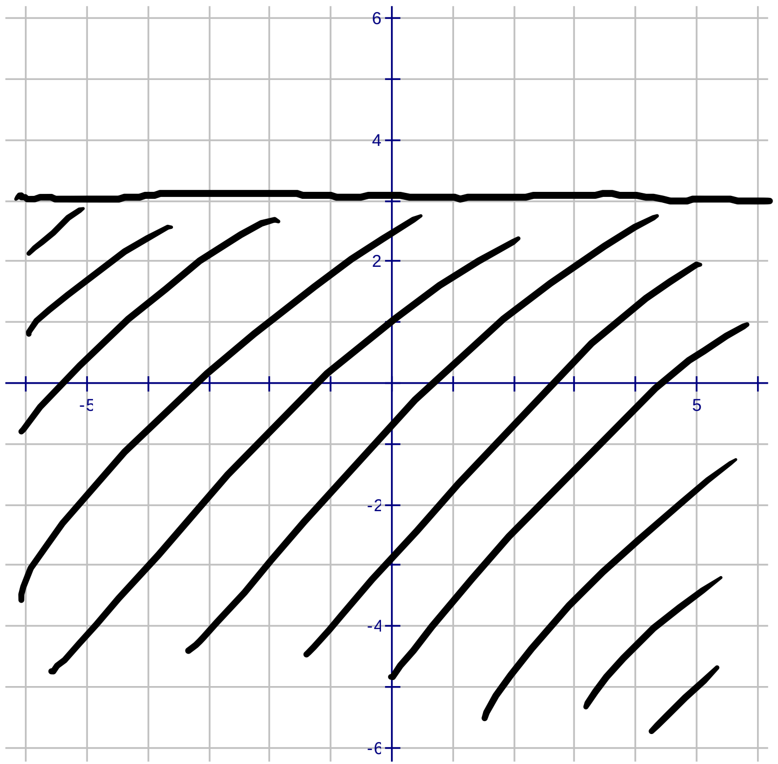
Test: $(0, 0)$

$0 - 0 < 2$

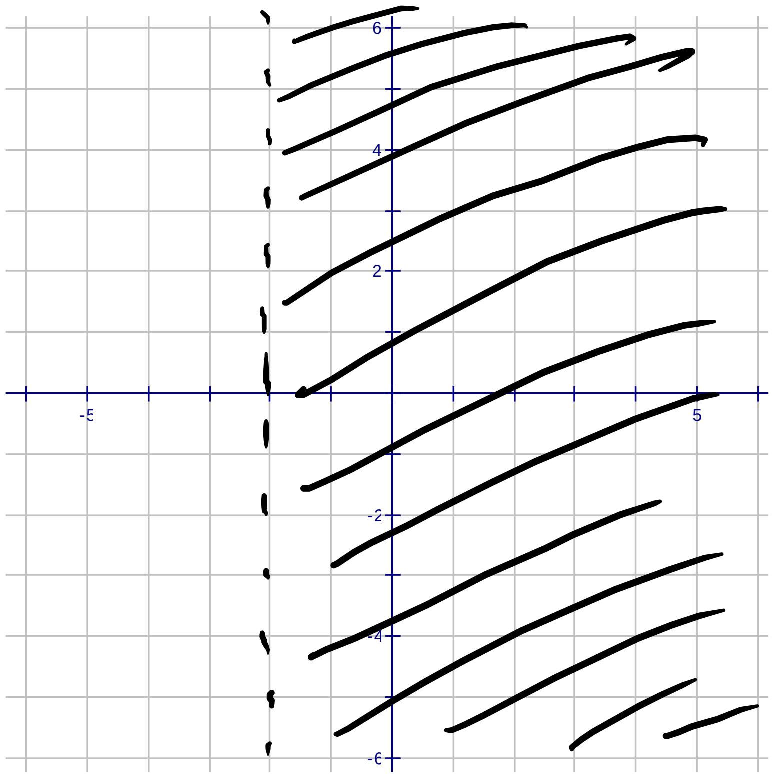
true



Ex. Sketch the graph of $y \leq 3$



Ex. Sketch the graph of $x > -2$



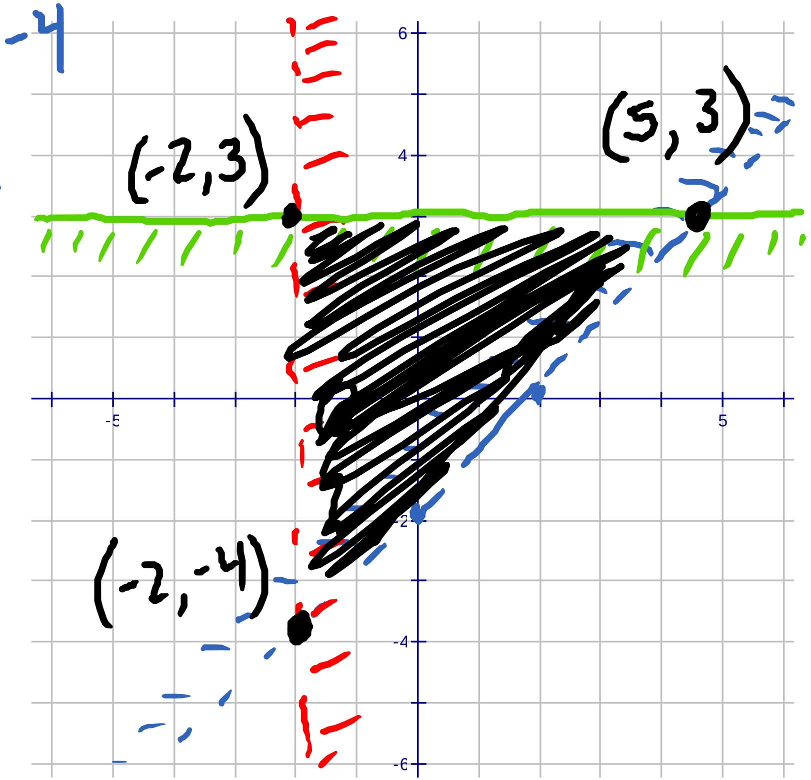
The solution of a system of inequalities is a graph of all points that satisfy each inequality

Ex. Sketch the graph, and label the vertices, of the solution set of the system

$$\begin{cases} x - y < 2 \\ x > -2 \\ y \leq 3 \end{cases}$$

$x=-2$: $-2-y=2$
$y=-4$

$y=3$: $x-3=2$
$x=5$



<u>Ex.</u> Sketch the solution set of the system

$$\begin{cases} x^2 - y \le 1 \\ -x + y \le 1 \end{cases}$$

$x^2 - y = 1$

$-y = -x^2 + 1$

$y = x^2 - 1$

<u>Test:</u> $(0,0)$

$0 - 0 \le 1$

true

$-x + y = 1$

$y = x + 1$

$x = 0 \rightarrow y = 1$

$x = 1 \rightarrow y = 2$

<u>Test:</u> $(0,0)$

$0 + 0 < 1$

true

$$\begin{cases} y = x^2 - 1 \\ y = x + 1 \end{cases}$$

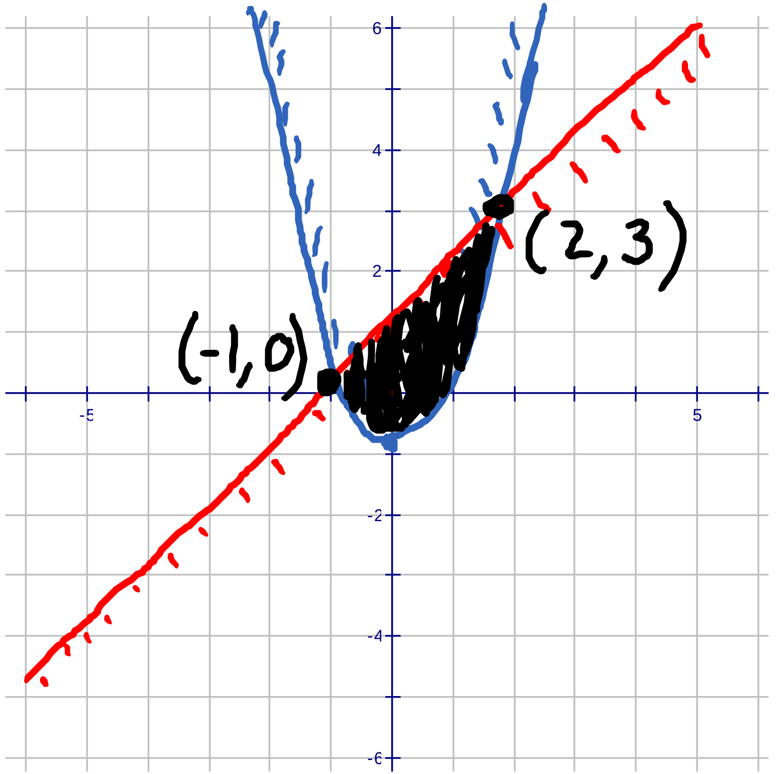
$x^2 - 1 = x + 1$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2 \qquad x = -1$

$y = 2 + 1 = 3 \qquad y = -1 + 1 = 0$



Ex. Sketch the solution set of the system

$$\begin{cases} x + y < 3 \\ x + 2y > 3 \end{cases}$$

$x + y = 3$

$x = 0 \rightarrow y = 3 \quad (0,3)$

$y = 0 \rightarrow x = 3 \quad (3,0)$

Test: $(0,0)$

$0 + 0 < 3$

true

$x + 2y = 3$

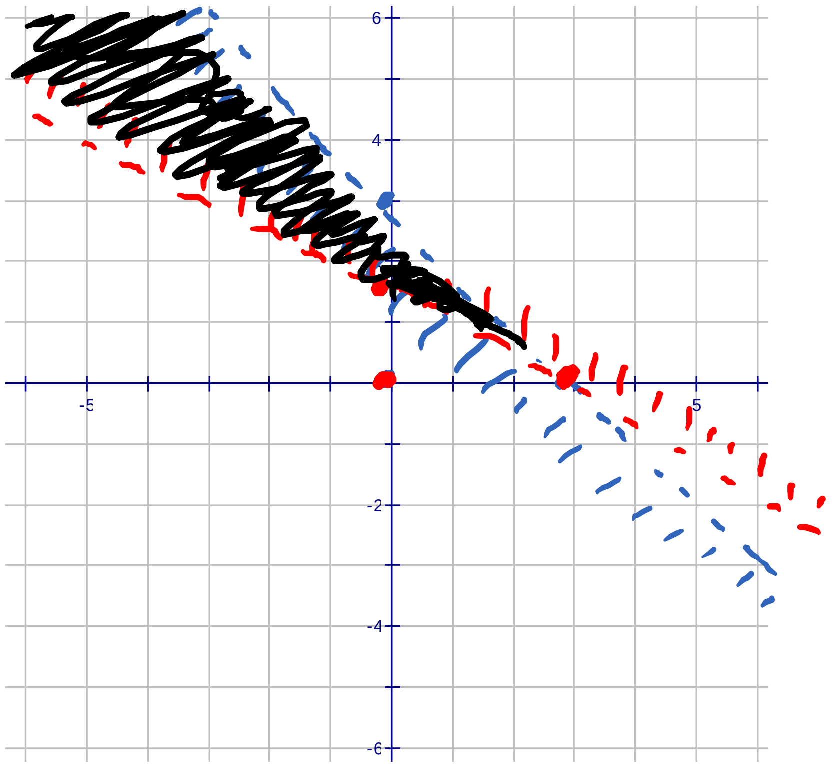
$x = 0 \rightarrow 2y = 3 \quad (0, \frac{3}{2})$

$y = \frac{3}{2}$

$y = 0 \rightarrow x = 3 \quad (3,0)$

Test: $(0,0)$

$0 + 0 > 3$ false



Ex. Sketch the solution set of the system

$$\begin{cases} x+y>3 \\ x+y<-1 \end{cases}$$

$x+y=3$

Test: $(0,0)$

$0+0>3$

false

$x+y=-1$

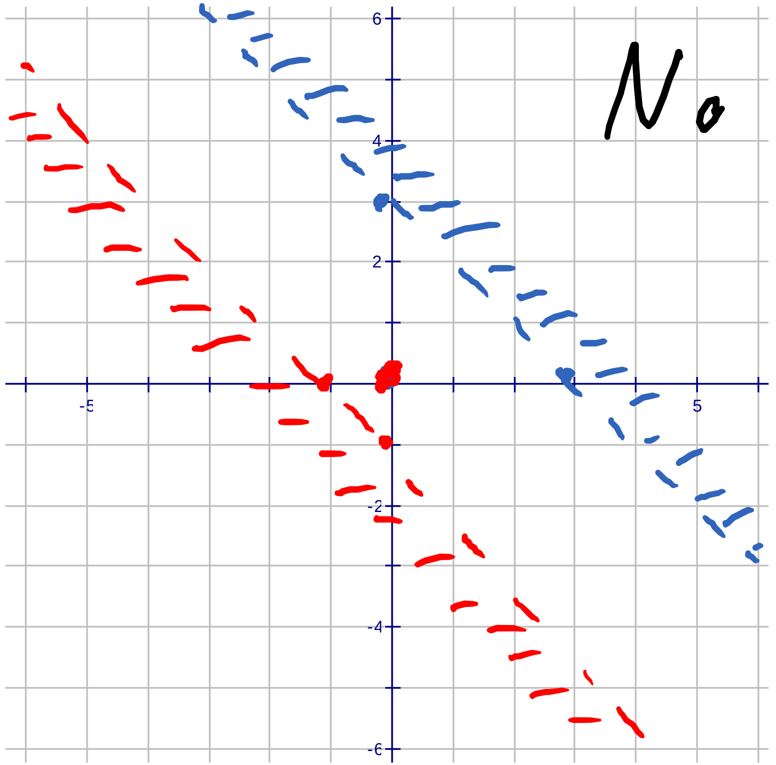
$x=0 \to y=-1 \quad (0,-1)$

$y=0 \to x=-1 \quad (-1,0)$

Test: $(0,0)$

$0+0<-1$

false



No solution

Ex. You plan to invest up to \$20,000 in two different accounts, with each account containing at least \$5000. Also, the amount in Account A should have at least twice the amount in Account B. Find and graph a system of inequalities that demonstrates this situation.

x: amt. in A

y: amt. in B

$$\begin{cases} x + y \le 20000 \\ x \ge 5000 \\ y \ge 5000 \\ x \ge 2y \end{cases}$$

$x + y = 20000$

Test: $(0,0)$

$0 + 0 < 20000$

$x = 2y$

$y = 5000 \to x = 10000$

$(10000, 5000)$

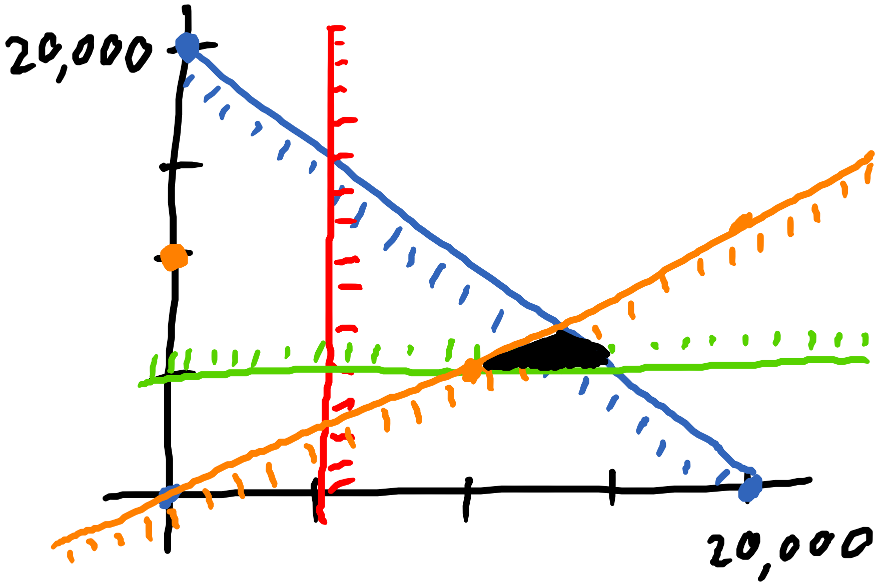
$y = 10000 \to x = 20000$

$(20000, 10000)$

Test: $(0, 10000)$

$0 \ge 2(10000)$

false



x: # cups of X
y: # cups of Y

Ex. The liquid portion of a diet must provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of drink *X* has 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of drink *Y* has 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up and graph a system of inequalities that shows how many cups of each drink would meet or exceed the minimum requirements.

$60x + 60y \geq 300$

$12x + 6y \geq 36$

$10x + 30y \geq 90$

$x \geq 0$

$y \geq 0$

$60x + 60y = 300$

$x + y = 5$ → $x=0, y=5$; → $y=0, x=5$

Test: $(0,0)$

$0+0 > 300$

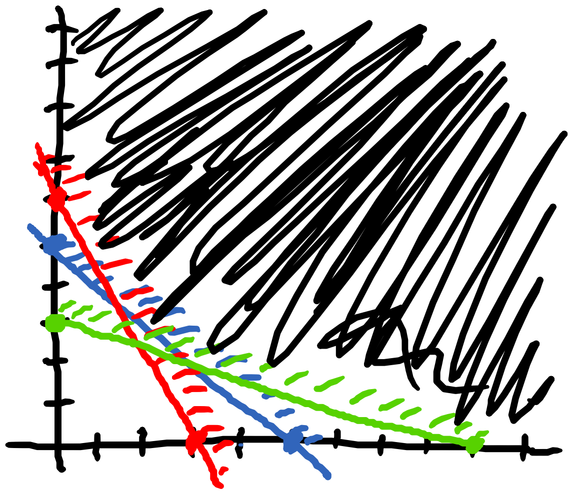
false

$12x+6y=36$

$2x + y = 6$ → $x=0, y=6$; → $y=0, 2x=6$, $x=3$

$10x+30y=90$

$x + 3y = 9$ → $x=0$, $3y=9$, $y=3$; → $y=0$, $x=9$



Test: $(0,0)$

$0+0 \geq 36$ false

Test $(0,0)$

$0+0 \geq 90$ false

Linear Programming

An optimization problem involves:

- a value to be maximized (or minimized), called the objective function
- restrictions on the variables in the objective function, called the constraints

The constraints are a system of inequalites like we saw earlier, and the solution graph shows all feasible solutions

The maximum and minimum values of the objective function will occur at a vertex of the shaded region

Ex. Find the maximum value of $z = 3x + 2y$ subject to the constraints

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + 2y \leq 4 \\ x - y \leq 1 \end{cases}$$

$(0,0)$: $z = 3(0) + 2(0) = 0$

$(0,2)$: $z = 3(0) + 2(2) = 4$

$(1,0)$: $z = 3(1) + 2(0) = 3$

$(2,1)$: $z = 3(2) + 2(1) = 8$

$x + 2y \leq 4 \rightarrow x + 2y = 4$

$x = 0 \rightarrow 2y = 4$

$y = 2$

$y = 0 \rightarrow x = 4$

Test: $(0,0)$

$0 + 0 < 4$

true

$x - y \leq 1$

$x - y = 1$

$x = 0 \rightarrow -y = 1$

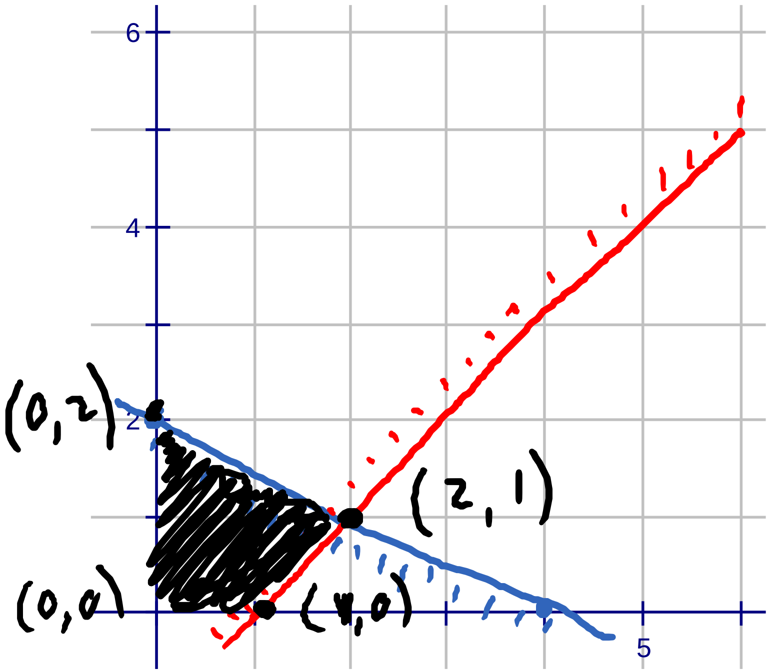
$y = -1$

$y = 0 \rightarrow x = 1$

Test: $(0,0)$

$0 - 0 < 1$

true



Ex. Find the minimum value of $z = 5x + 7y$ subject to the constraints

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 3y \geq 6 \\ 3x - y \leq 15 \\ -x + y \leq 4 \\ 2x + 5y \leq 27 \end{cases}$$

$2x + 3y = 6$

$x = 0 \rightarrow 3y = 6$

$y = 2$

$y = 0 \rightarrow 2x = 6$

$x = 3$

Test: $(0,0)$

$0 + 0 \geq 6$

false

$3x - y = 15$

$y = 3x - 15$

$x = 5 \rightarrow y = 0$

$x = 6 \rightarrow y = 3$

Test: $(0,0)$

$0 - 0 \leq 15$

true

$-x + y = 4$

$y = x + 4$

$x = 0 \rightarrow y = 4$

$x = 1 \rightarrow y = 5$

Test: $(0,0)$

$0 + 0 \leq 4$

true

$2x + 5y = 27$

$x = 0 \rightarrow 5y = 27$

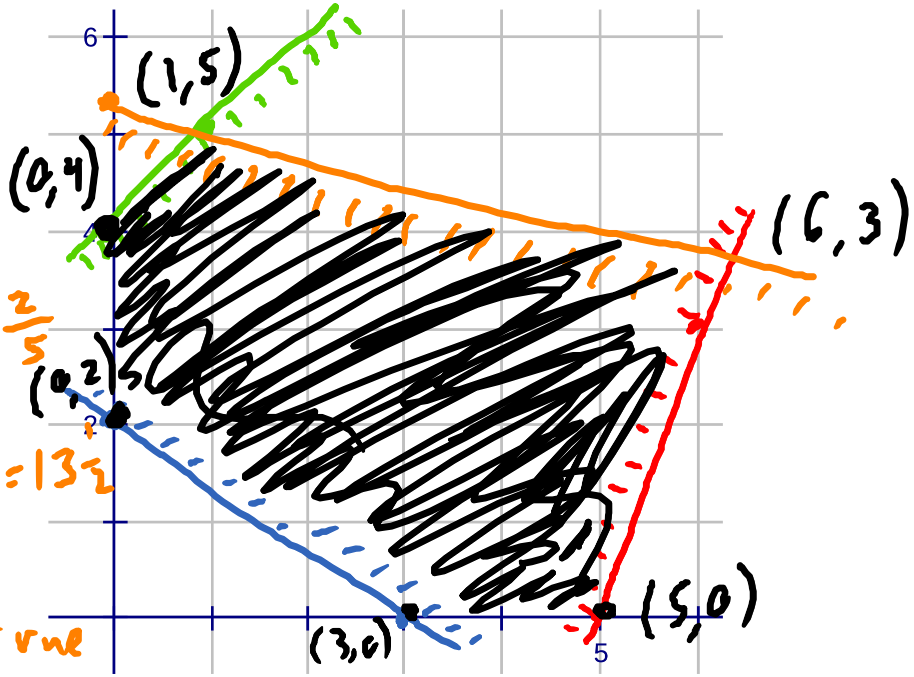
$y = \frac{27}{5} = 5\frac{2}{5}$

$y = 0 \rightarrow 2x = 27$

$x = \frac{27}{2} = 13\frac{1}{2}$

Test: $(0,0)$

$0 + 0 \leq 27$ true



$$2x+5y=27 \longrightarrow 2x+5y=27$$
$$-x+y=4 \xrightarrow{x\,5} 5x-5y=-20$$
$$7x=7$$
$$x=1$$
$$-1+y=4$$
$$y=5$$

$$2x+5y=27 \longrightarrow 2x+5y=27$$
$$3x-y=15 \xrightarrow{x\,5} 15x-5y=75$$
$$17x=102$$
$$x=6$$
$$3(6)-y=15$$
$$18-y=15$$
$$y=3$$

$$\boxed{z=5x+7y}$$

$(1,5)$: $z=5(1)+7(5)=40$

$(6,3)$: $z=5(6)+7(3)=51$

$(3,0)$: $z=5(3)+7(0)=15$

$(5,0)$: $z=5(5)+7(0)=25$

$(0,2)$: $z=5(0)+7(2)=14$

$(0,4)$: $z=5(0)+7(4)=28$

Ex. A manufacturer is making two types of candy, chocolate creams that net \$1.50 and chocolate nuts that net \$2.00, and wants to maximize profit. Market tests and available resources create the following constraints:

- The combined number of boxes should not exceed 1200
- The demand for chocolate nuts is no more than half the demand for chocolate creams
- The number of chocolate cream boxes is no more than 600 plus three times the number of chocolate nuts

Ex. The liquid portion of a diet must provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of drink *X* has 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of drink *Y* has 60 calories, 6 units of vitamin A, and 30 units of vitamin C. If drink *X* costs $0.12 and drink *Y* costs $0.15, how many cups of each drink must be consumed to obtain optimal cost and still meet the minimum requirements?

x: # cups of X
y: # cups of Y

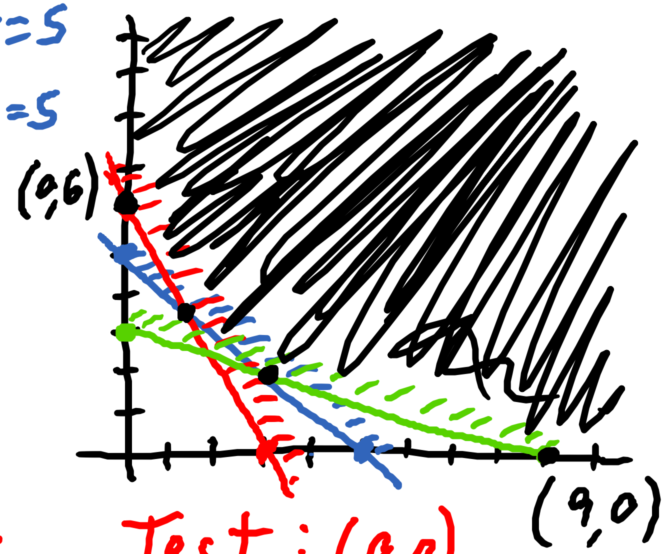
$60x + 60y \geq 300$
$12x + 6y \geq 36$
$10x + 30y \geq 90$
$x \geq 0$
$y \geq 0$

$C = .12x + .15y$
Make minimum

$60x + 60y = 300$
$x + y = 5 \rightarrow x = 0, y = 5$
$\rightarrow y = 0, x = 5$
Test: $(0,0)$
$0 + 0 \geq 300$
false

$12x + 6y = 36$
$2x + y = 6 \rightarrow x = 0, y = 6$
$\rightarrow y = 0, 2x = 6$
$x = 3$

$10x + 30y = 90$
$x + 3y = 9 \rightarrow x = 0, 3y = 9$
$y = 3$
$\rightarrow y = 0, x = 9$



Test: $(0,0)$
$0 + 0 \geq 36$ false

Test $(0,0)$
$0 + 0 \geq 90$ false

$$x+y=5 \xrightarrow{x-1} -x-y=-5$$
$$2x+y=6 \longrightarrow 2x+y=6$$
$$x = 1$$

$$1+y=5$$
$$y=4$$

$$x+y=5 \xrightarrow{x-1} -x-y=-5$$
$$x+3y=9 \longrightarrow x+3y=9$$
$$2y=4$$
$$y=2$$

$$x+2=5$$
$$x=3$$

$$C=.12x+.15y$$

$(1,4): C=.12(1)+.15(4)=.72$

$(3,2): C=.12(3)+.15(2)=.66$

$(0,6): C=.12(0)+.15(6)=.9$

$(9,0): C=.12(9)+.15(0)=1.08$