First part is out of 15 pts.
Second part is out of 90 pts.
Total points possible is 105 pts.
→ Grade is out of 100 pts.

Using Matrices

A matrix is a rectangular array that can help us to streamline the solving of a system of equations

$$\begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & 9 \end{bmatrix}$$

The order of this matrix is 2×3

If the number of rows and columns is equal, we say that the matrix is square

$$\begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & 9 \\ -4 & 8 & 6 \end{bmatrix}$$

The 5 is entry a_{12} because it is in the 1st row and 2nd column

In a square matrix, entries a_{11}, a_{22}, etc. are called the <u>main diagonal</u>

By changing a system of equations into a matrix (<u>augmented matrix</u>), we can make it easier to work with

$$\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x \qquad\quad + 4z = 6 \end{cases} \Rightarrow \begin{array}{c} \begin{array}{cccc} x & y & z & = \end{array} \\ \left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right] \end{array}$$

The left side is called the <u>coefficient matrix</u>

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & 4 \end{bmatrix}$$

Ex. Write the augmented matrix of the system

$$\begin{cases} x + 3y - w = 9 \\ -y + 4z + 2w = -2 \\ x - 5y - 6w = 0 \\ 2x + 4y - 3z = 4 \end{cases} \Rightarrow \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 9 \\ 0 & -1 & 4 & 2 & -2 \\ 1 & -5 & 0 & -6 & 0 \\ 2 & 4 & -3 & 0 & 4 \end{array}\right] \\ \begin{array}{ccccc} x & y & z & w & \end{array} \end{array}$$

The same row operations can be used:

- Rows can be switched

$$\left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Rightarrow} \left[\begin{array}{ccc:c} -1 & 3 & -1 & -3 \\ 1 & -4 & 3 & 5 \\ 2 & 0 & 4 & 6 \end{array}\right]$$

The same row operations can be used:

- Rows can be multiplied by a constant

$$\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right] \underset{R_3 \to 4R_3}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 8 & 0 & 16 & 24 \end{array}\right]$$

The same row operations can be used:

- Rows can be replaced with the sum of two rows

$$\left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right] \underset{R_3 \to R_2 + R_3}{\Rightarrow} \left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 1 & 3 & 3 & 3 \end{array}\right]$$

We are going to use row operations to put a matrix into row-echelon form (REF)

- Any row with all zeroes is at the bottom
- If the row isn't all zeroes, the first nonzero entry from the left is a 1
- Each leading 1 has zeroes below it

If each leading 1 has zeroes above and below it, we say the matrix is in reduced row-echelon form (RREF)

These are in row-echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

These are in reduced row-echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

These are not in row-echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Fix $R_2 \longleftrightarrow R_3$ Fix $R_2 \to \frac{1}{2} R_2$

To put a matrix into row-echelon form:

- Make the upper-left entry one, then <u>use it</u> to get zeroes below it
- Move down and right, repeating the process

Putting an augmented matrix into row-echelon form will make it easier to solve

Ex. Put into row-echelon form

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_2 + R_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{2}R_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Ex. Solve

$$\begin{cases} y+z-2w=-3 \\ x+2y-z=2 \\ 2x+4y+z-3w=-2 \\ x-4y-7z-w=-19 \end{cases} \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \xRightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_1 - R_2 \\ \Longrightarrow \\ R_3 \to -2R_1 + R_3 \\ R_4 \to R_1 - R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 7 & 7 & -1 & 18 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_3 \to 2R_2 + R_3 \\ R_4 \to -7R_2 + R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & 13 & 39 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_3 \to \frac{1}{3}R_3 \\ R_4 \to \frac{1}{13}R_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & w & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \end{array}$$

$$\Longrightarrow \begin{array}{l} x+3y-2w=-1 \longrightarrow x+3(-2)-2(3)=-1 \longrightarrow x-6-6=-1 \longrightarrow \boxed{x=11} \\ y+z-2w=-3 \longrightarrow y+5-2(3)=-3 \longrightarrow y+5-6=-3 \quad \boxed{y=-2} \\ z-w=2 \longrightarrow z-3=2 \quad \boxed{z=5} \\ \boxed{w=3} \end{array}$$

Ex. Solve

$$\begin{cases} x - y + 2z = 4 \\ x \qquad + z = 6 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 1 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{array}\right] \xRightarrow[\substack{R_3 \to -2R_1 + R_3 \\ R_4 \to -3R_1 + R_4}]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{array}\right]$$

$$\xRightarrow[\substack{R_3 \to R_2 + R_3 \\ R_4 \to -5R_2 + R_4}]{} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -21 \end{array}\right] \xRightarrow[R_3 \longleftrightarrow R_4]{} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & -21 \\ 0 & 0 & 0 & -2 \end{array}\right] \longrightarrow 0 = -2$$

No solution

We've been using Gaussian elimination

- a matrix in row echelon form may be different depending on the order that you do the row operations, but the answer will be the same

Using Gauss-Jordan elimination means to put the augmented matrix into reduced-row echelon form

- the reduced-row echelon form will be unique

Ex. Solve using Gauss-Jordan elimination

$$\begin{cases} x-2y+3z=9 \\ -x+3y \qquad = -4 \\ 2x-5y+5z=17 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow[R_3 \to -2R_1+R_3]{R_2 \to R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_2+R_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{2}R_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 \to -3R_3+R_2]{R_1 \to -3R_3+R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \to 2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longrightarrow \begin{array}{l} x=1 \\ y=-1 \\ z=2 \end{array}$$

Ex. Solve

$$\begin{cases} 2x+4y-2z=0 \\ 3x+5y \qquad\quad =1 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{array}\right] \Longrightarrow \begin{array}{l} x+5z=2 \\ y-3z=-1 \end{array} \Longrightarrow \begin{array}{l} x=-5z+2 \\ y=3z-1 \end{array}$$

$$\boxed{\text{Let } z=k} \Longrightarrow \begin{array}{l} x=-5k+2 \\ y=3k-1 \end{array} \qquad \boxed{(-5k+2,\ 3k-1,\ k)}$$

Operations with Matrices

Two matrices are equal if they have the same order and if the corresponding entries are equal

Adding and subtracting matrices means performing the operations on corresponding entries

- The matrices must have the same order, and the result will also have that order

Ex.

a. $$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

2×2 $\quad$ 2×2 $\quad$ 2×2

b. $$\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 1 & 4 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 \\ 1 & 0 & 8 \end{bmatrix}$$

2×3 $\quad$ 2×3 $\quad$ 2×3

Scalar multiplication means multiplying a matrix by a constant

- We do this by multiplying each entry by the constant

Ex. Let $A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

a. $3A = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

b. $3A - B$

Ex. Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$$

$$3X + A = B$$
$$3X = B - A$$
$$X = \frac{1}{3}(B - A)$$

$$X = \frac{1}{3}(B - A)$$
$$= \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$
$$= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -4/3 & 2 \\ 2/3 & -2/3 \end{bmatrix}$$

When multiplying two matrices, we take a row from the first matrix and multiply it by a column from the second matrix

The orders have to match up:

$$\underset{4\times 3}{A} \times \underset{3\times 7}{B} = \underset{4\times 7}{AB}$$

Notice that the resulting order would change if the order of multiplication changes

- Switching the matrices changes the answer

Ex. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$

$$(1)(-2) + (0)(1) + (3)(-1) = -5$$

$$(1)(4) + (0)(0) + (3)(1) = 7$$

$$(2)(-2) + (-1)(1) + (-2)(-1) = -3$$

$$(2)(4) + (-1)(0) + (-2)(1) = 6$$

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 7 & 35 \end{bmatrix}$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

Ex. $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 1 \qquad 2 \times 1$

The <u>identity matrix</u> is a square matrix that has ones on its main diagonal and zeroes as every other entry

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying any matrix by I results in the same matrix

$$AI = IA = A$$

A system of equations can be expressed as a matrix equation

$$\begin{cases} x - 2y + z = -4 \\ y + 2z = 4 \\ 2x + 3y - 2z = 2 \end{cases} \Rightarrow \; AX = B$$

where $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$