Inverse Matrices

2 and ½ are multiplicative inverses because

$$2 \times \tfrac{1}{2} = 1$$

The inverse of matrix A is written A^{-1}, and

$$AA^{-1} = I \text{ and } A^{-1}A = I$$

where I is the identity matrix

Ex. Show that $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ are inverses

$$A \cdot B = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

If a matrix has an inverse, we say that it is invertible

- Otherwise, we say that it is singular
- Only square matrices can be invertible

One way to find an inverse matrix is by using the row operations

- Create the matrix $[A \mid I]$
- Perform row operations to make the left side into I
- The result will be $[I \mid A^{-1}]$

Ex. Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to -6R_1 + R_3]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to -4R_2 + R_3]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + R_3]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$\therefore$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

For a 2×2 matrix, there's a quicker way

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called the <u>determinant</u> of the matrix...we'll look at that later

Ex. Find the inverse

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{6-2}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

b. $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

$$B^{-1} = \frac{1}{6-6}\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$$

not invertible

To solve the equation $ax = b$, we multiply by the multiplicative inverse $\frac{1}{a}$:

$$ax = b$$

$$\frac{1}{a}ax = \frac{1}{a}b$$

$$x = \frac{b}{a}$$

To solve a matrix equation, we do the same

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

Ex. Solve the system $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases} \Longrightarrow \begin{bmatrix} 3 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -2 \\ 4 \end{bmatrix}}_{B}$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9-20}\begin{bmatrix} 3 & -4 \\ -5 & 3 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} 3 & -4 \\ -5 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{-11}\begin{bmatrix} 3 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -22 \\ 22 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$x = 2 \quad y = -2$

Determinant of a Matrix

The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We can only find the determinant of a square matrix

Ex. Find the determinant

a. $\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 4 - (-3) = 7$

b. $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0$

c. $\begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0 - 3 = -3$

To find the determinant of a larger matrix, we will use minors and cofactors

The minor of entry a_{ij}, written M_{ij}, is the determinant of the matrix found by removing row i and column j of matrix A

The cofactor of entry a_{ij}, written C_{ij}, is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Ex. Find some minors and cofactors of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$

$a_{21} = 3$ $\quad M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$ $\quad C_{21} = (-1)^{2+1}(2) = -2$

$a_{23} = 2$ $\quad M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} = -8$ $\quad C_{23} = (-1)^{2+3}(-8) = 8$

$a_{31} = 4$ $\quad M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$ $\quad C_{31} = (-1)^{3+1}(5) = 5$

The determinant of a square matrix is the sum of the entries of any row (or column) multiplied by the corresponding cofactors

Ex. Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$

a	M	C	a · C
$a_{11} = 0$	$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1$	$C_{11} = (-1)^{1+1}(-1) = -1$	$(0)(-1) = 0$
$a_{21} = 3$	$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$	$C_{21} = (-1)^{2+1}(2) = -2$	$(3)(-2) = -6$
$a_{31} = 4$	$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$	$C_{31} = (-1)^{3+1}(5) = 5$	$(4)(5) = 20$

$\det A = 14$

Ex. Find the determinant of $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

↑ (column 3)

a	M	C	$a \cdot C$
$a_{13} = 3$	$M_{13} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$	$C_{13} = (-1)^{1+3}(1) = 1$	$(3)(1) = 3$
$a_{23} = 0$	—	—	— $= 0$
$a_{33} = 4$	$M_{33} = \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} = 0$	$C_{33} = (-1)^{3+3}(0) = 0$	$(4)(0) = 0$

$\boxed{\det A = 3}$

Ex. Find the determinant of $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & 2 \end{bmatrix}$

$$3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix}$$

$$= 3\left[0(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + 2(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} + 3(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right]$$

$$= 3\left(0 + 2(-2-6) - 3(-4-3)\right) = 3(-16+21) = \boxed{15}$$

Applications of Matrices

It is possible to solve a system of equations by finding a bunch of determinants

Cramer's Rule

Let D be the matrix of coefficents on x and y

To find the first variable (x), create the matrix D_x where the first column of D is replaced by the constants of the system

$$x = \frac{|D_x|}{|D|}$$

Repeat to find the other variables

Ex. Solve $\begin{cases} 4x - 2y = 10 \\ 3x - 5y = 11 \end{cases}$

The area of a triangle with vertices (x_1,y_1), (x_2,y_2), and (x_3,y_3) is

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Ex. Find the area of the triangle whose vertices are (1,0), (2,2), and (4,3).

$$A = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[0 \cdot \text{———} + 2(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} + 3(-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\right]$$

$$= \frac{1}{2}\left[2(1-4) - 3(1-2)\right] = \frac{1}{2}(-6+3) = -\frac{3}{2}$$

$$A = \frac{3}{2}$$

Matrices can be used to encode messages:

Each letter is assigned a number
(_ = 0, A = 1, B = 2, etc.)

Group letters into 1×3 matrices

Multiply each matrix by an invertible 3×3 matrix to get a coded matrix

Decode by multiplying by the inverse matrix

13 5 5, 20 0 13, 5 0 13, 15 14 4, 1 25 0

Ex. Encode the message MEET ME MONDAY using the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$$

Ex. $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$ was used to create the cryptogram

$$\begin{bmatrix} -1 & -7 & 43 \end{bmatrix} \begin{bmatrix} 11 & -24 & 19 \end{bmatrix}$$

Find the original message.