Inverse Matrices

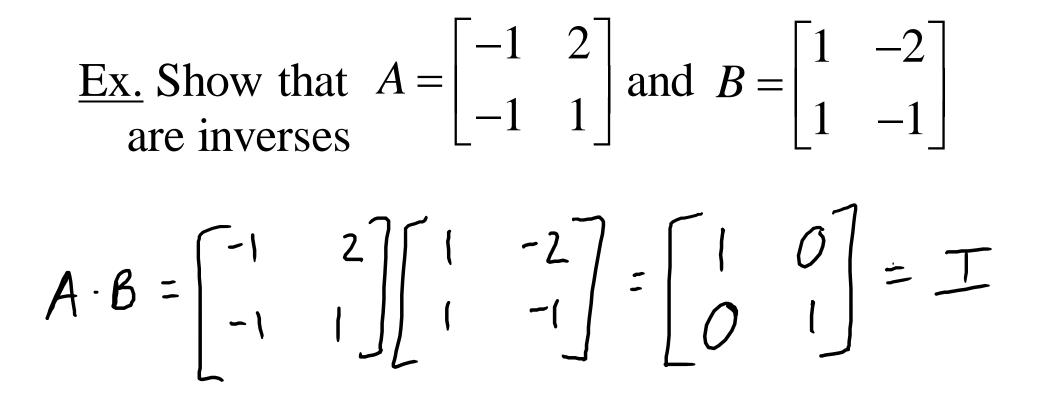
2 and ¹/₂ are multiplicative inverses because

$$2 \times \frac{1}{2} = 1$$

The <u>inverse</u> of matrix A is written A^{-1} , and

$$AA^{-1} = I$$
 and $A^{-1}A = I$

where *I* is the identity matrix



If a matrix has an inverse, we say that it is <u>invertible</u>

- Otherwise, we say that it is <u>singular</u>
- Only square matrices can be invertible

One way to find an inverse matrix is by using the row operations

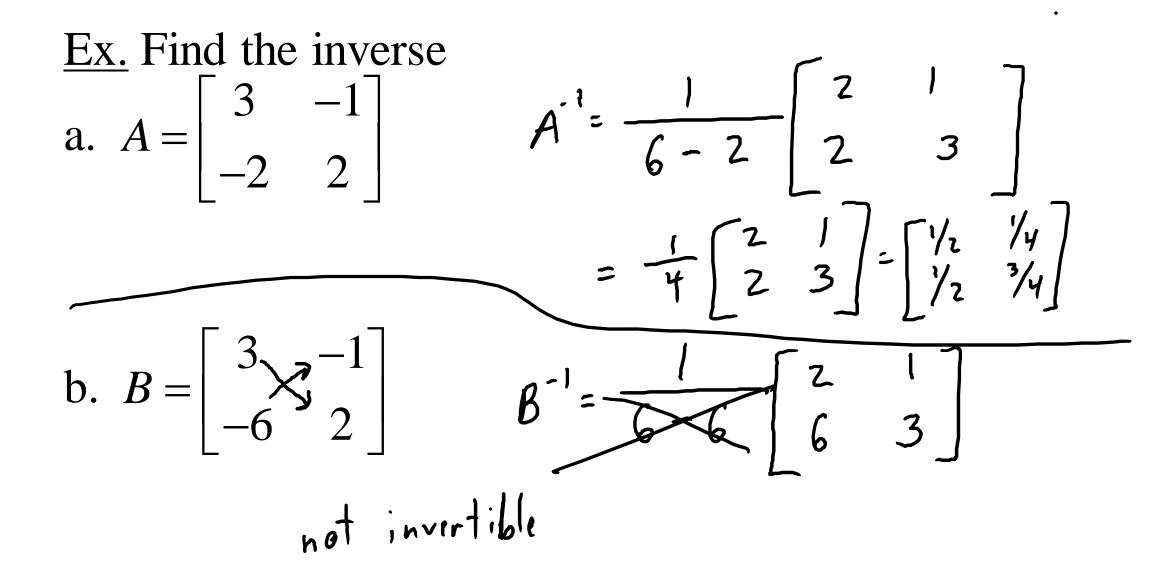
- Create the matrix [*A* | *I*]
- Perform row operations to make the left side into *I*
- The result will be $[I | A^{-1}]$

$$\underbrace{\operatorname{Ex. Find the inverse of } A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}} \\
\begin{pmatrix} \textcircled{1} & -1 & 0 \\ 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 0 & 1 \\ R_{3} + R_{1} + R_{3} \end{bmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ R_{3} + R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ R_{3} + R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ R_{3} + R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ R_{3} + R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} + R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} + R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} + R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} + R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} + R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} - R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} - R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \\ R_{3} - R_{3} + R_{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ R_{3} - R_{3} + R_{3} \\ R_{3} - R_{3} \\ R_{3} \\ R_{3} - R_{3} \\ R_{3} \\ R_{3} - R_{3} \\ R_{3}$$

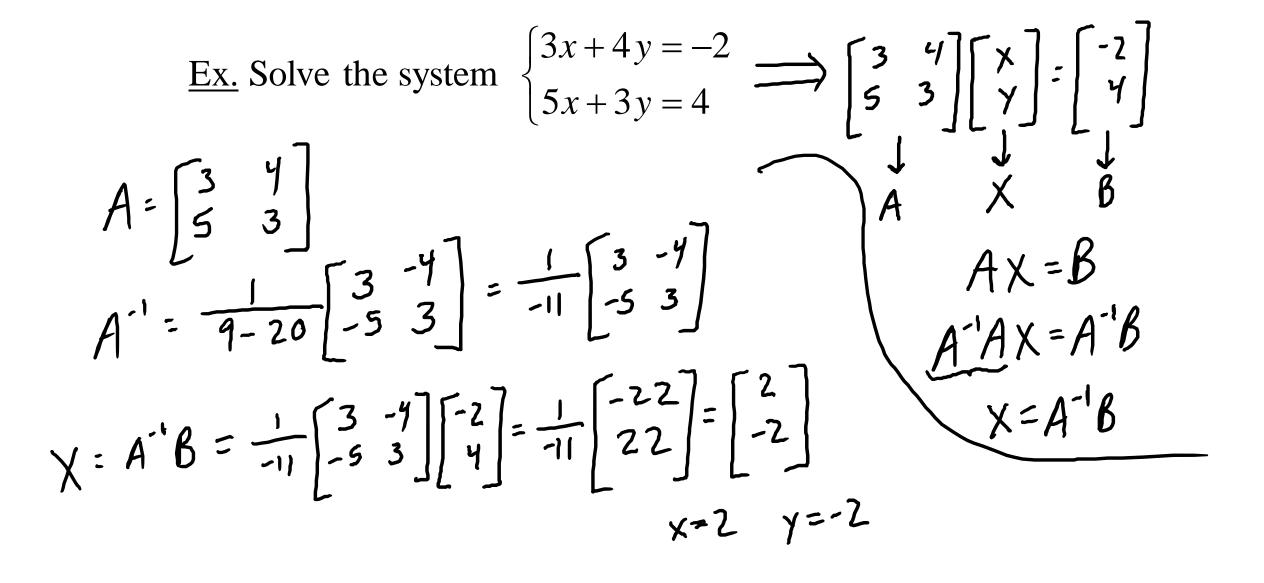
For a 2×2 matrix, there's a quicker way

If
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
, then $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The quantity ad - bc is called the <u>determinant</u> of the matrix...we'll look at that later



To solve the equation ax = b, we multiply by the multiplicative inverse $\frac{1}{a}$: ax = b $\frac{1}{a}ax = \frac{1}{a}b$ $x = \frac{b}{a}$ To solve a matrix equation, we do the same AX = B $A^{-1}AX = A^{-1}B$ $X = A^{-1}B$



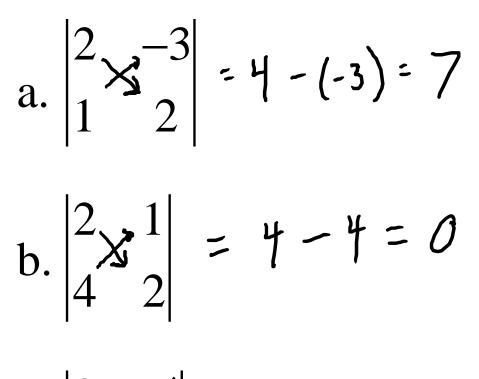
Determinant of a Matrix
The determinant of the matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We can only find the determinant of a square matrix

Ex. Find the determinant



c.
$$\begin{vmatrix} 0 \\ x \\ 2 \\ 4 \end{vmatrix}^{\frac{3}{2}} = 0 - 3 = -3$$

To find the determinant of a larger matrix, we will use minors and cofactors

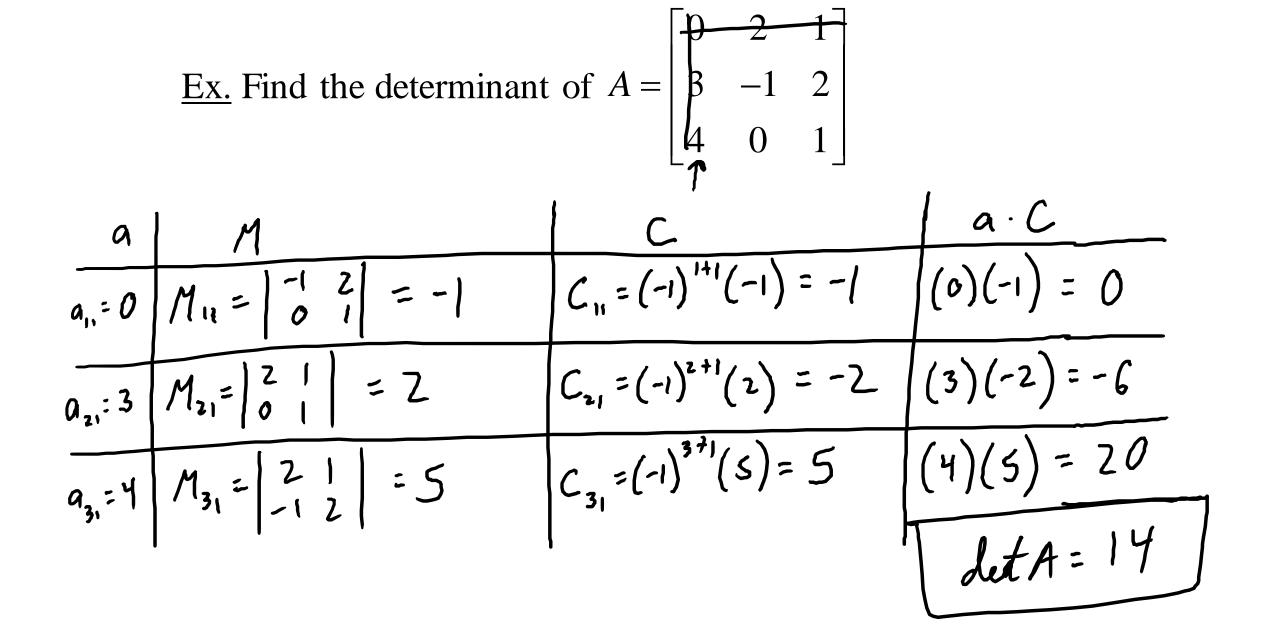
The <u>minor</u> of entry a_{ij} , written M_{ij} , is the determinant of the matrix found by removing row *i* and column *j* of matrix *A*

The <u>cofactor</u> of entry a_{ij} , written C_{ij} , is $C_{ij} = (-1)^{i+j} M_{ij}$

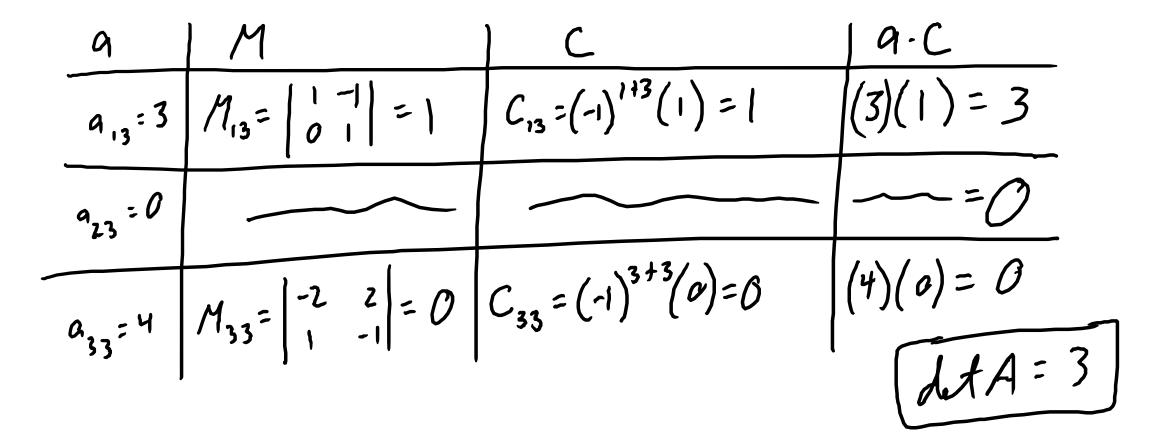
Ex. Find some minors and cofactors of
$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & -1 \end{bmatrix}$$

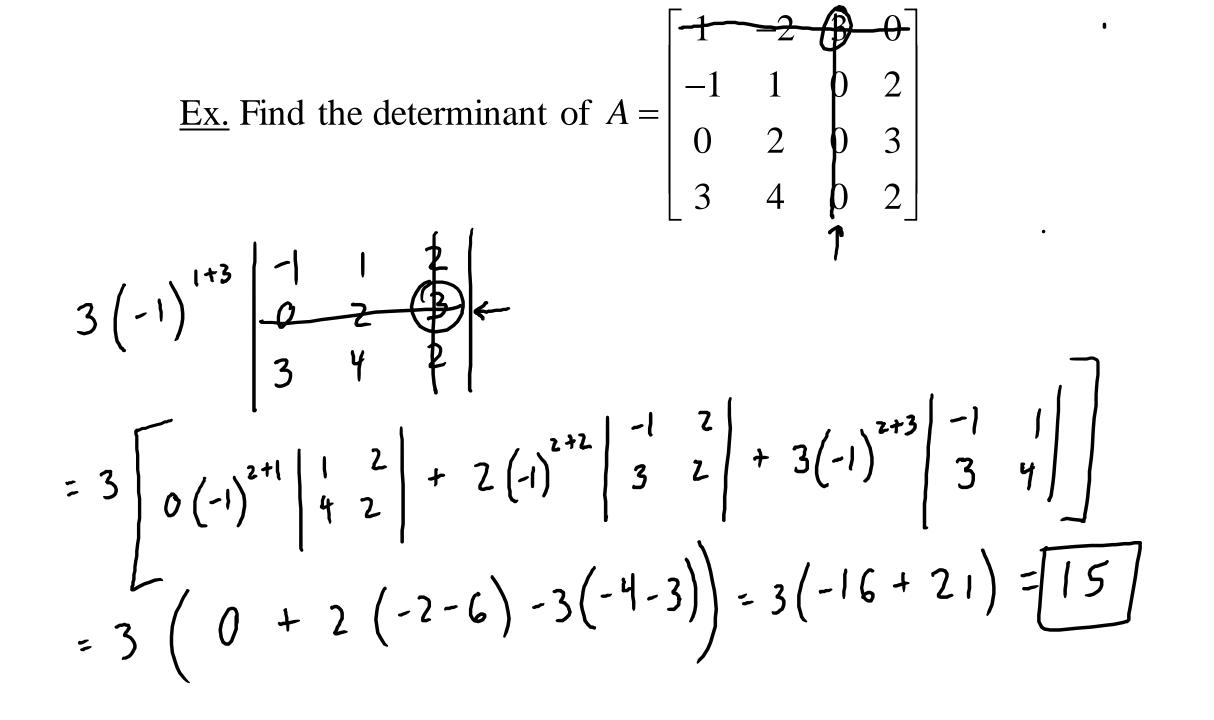
 $a_{21} = 3$ $M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$ $C_{21} = (-1)^{2+1}(2) = -2$
 $a_{23} = 2$ $M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} = -8$ $C_{23} = (-1)^{2+3}(-8) = 8$
 $a_{31} = 4$ $M_{3i} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$ $C_{51} = (-1)^{3+1}(5) = 5$

The determinant of a square matrix is the sum of the entries of any row (or column) multiplied by the corresponding cofactors



Ex. Find the determinant of
$$A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$





Applications of Matrices

It is possible to solve a system of equations by finding a bunch of determinants

<u>Cramer's Rule</u> Let *D* be the matrix of coefficients on *x* and *y*

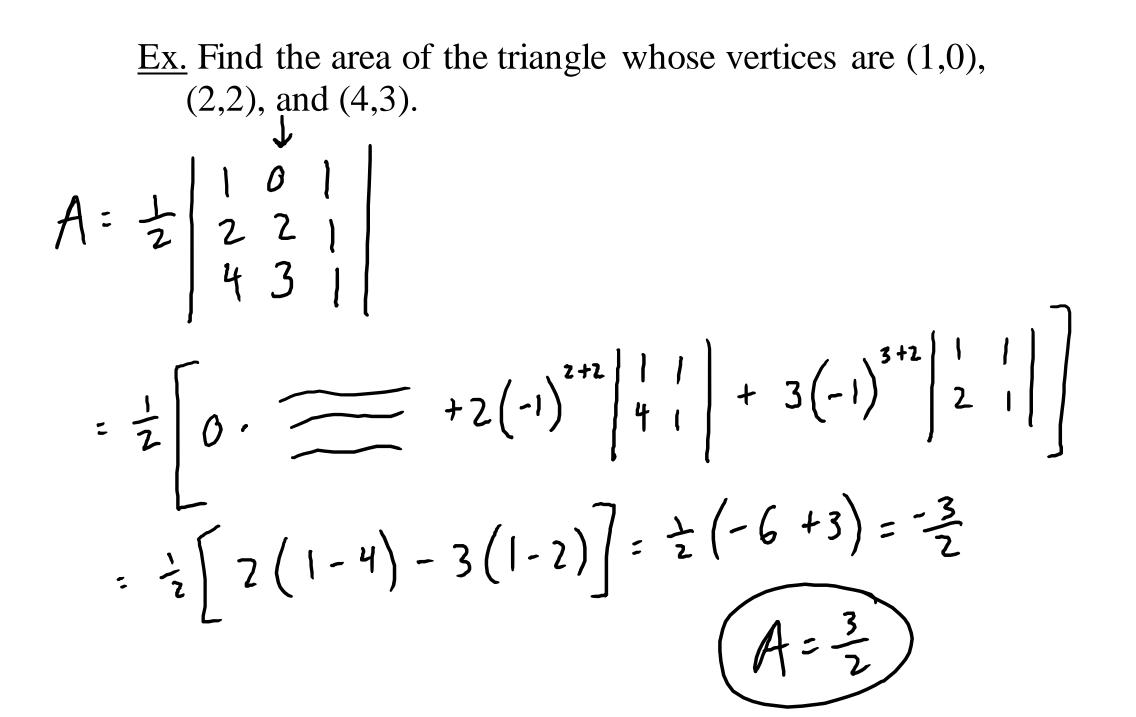
To find the first variable (*x*), create the matrix D_x where the first column of *D* is replaced by the constants of the system

$$x = \frac{\left|D_x\right|}{\left|D\right|}$$

Repeat to find the other variables

Ex. Solve
$$\begin{cases} 4x - 2y = 10\\ 3x - 5y = 11 \end{cases}$$

The area of a triangle with vertices (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) is $A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$



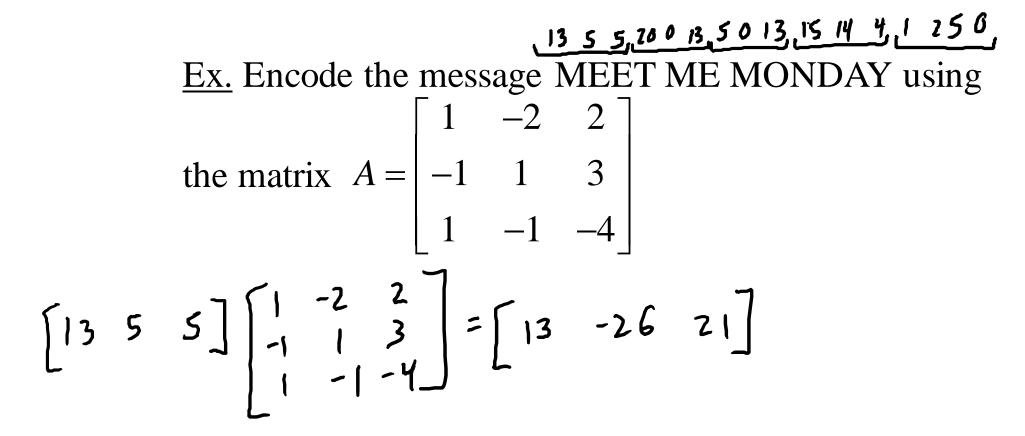
Matrices can be used to encode messages:

Each letter is assigned a number $(_=0, A=1, B=2, etc.)$

Group letters into 1×3 matrices

Multiply each matrix by an invertible 3×3 matrix to get a <u>coded matrix</u>

Decode by multiplying by the inverse matrix



[13 -26 2]

Ex.
$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$
 was used to create the cryptogram
 $\begin{bmatrix} -1 & -7 & 43 \end{bmatrix} \begin{bmatrix} 11 & -24 & 19 \end{bmatrix}$
Find the original message.