Series and Sequences

An infinite sequence is an unending list of numbers that follow a pattern. The terms of the sequence are written $a_1, a_2, a_3, \ldots, a_n, \ldots$

If the list ends, we call it a finite sequence.

Ex. Write the first four terms of the sequence:

a) $a_n = 3n - 2$

$a_1 = 3(1) - 2 = 1$

$a_2 = 3(2) - 2 = 4$

$a_3 = 3(3) - 2 = 7$

$a_4 = 3(4) - 2 = 10$

$1, 4, 7, 10, \ldots$

b) $a_n = 3 + (-1)^n$

alternates signs

$a_1 = 3 + (-1)^1 = 2$

$a_2 = 3 + (-1)^2 = 4$

$a_3 = 3 + (-1)^3 = 2$

$a_4 = 3 + (-1)^4 = 4$

$2, 4, 2, 4, \ldots$

$\underline{\text{Ex.}}$ Write the first four terms of the sequence

$$a_n = \frac{(-1)^n}{2n-1}$$

$(-1)^n \rightarrow$ alternates signs

$2n-1 \rightarrow$ odd numbers

$2n \rightarrow$ even numbers

$$a_1 = \frac{(-1)^1}{2(1)-1} = \frac{-1}{1}$$

$$a_2 = \frac{(-1)^2}{2(2)-1} = \frac{1}{3}$$

$$a_3 = \frac{(-1)^3}{2(3)-1} = \frac{-1}{5}$$

$$a_4 = \frac{(-1)^4}{2(4)-1} = \frac{1}{7}$$

$$-\frac{1}{1}, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, \ldots$$

Ex. Write an expression for a_n:

a) 1, 3, 5, 7,...

$a_n = 2n - 1$

b) 1, -4, 9, -16,...

$n=1$ $n=2$ $n=3$ $n=4$

$a_n = (-1)^{n+1} n^2$

A sequence is <u>recursive</u> if each term is defined by one or more previous terms

previous term in sequence

<u>Ex.</u> Find the first five terms of the recursive sequence defined by $a_1 = 3$, $\boxed{a_{n+1} = 2\boxed{a_n} - 5}$

$\boxed{3, 1, -3, -11, -27, \ldots}$

$a_1 = 3$

$n=1:$ $a_2 = 2a_1 - 5 = 2(3) - 5 = 1$

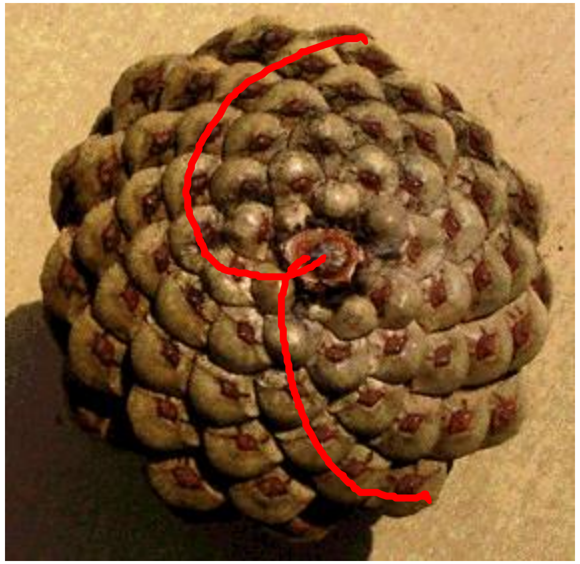
$a_3 = 2(1) - 5 = -3$

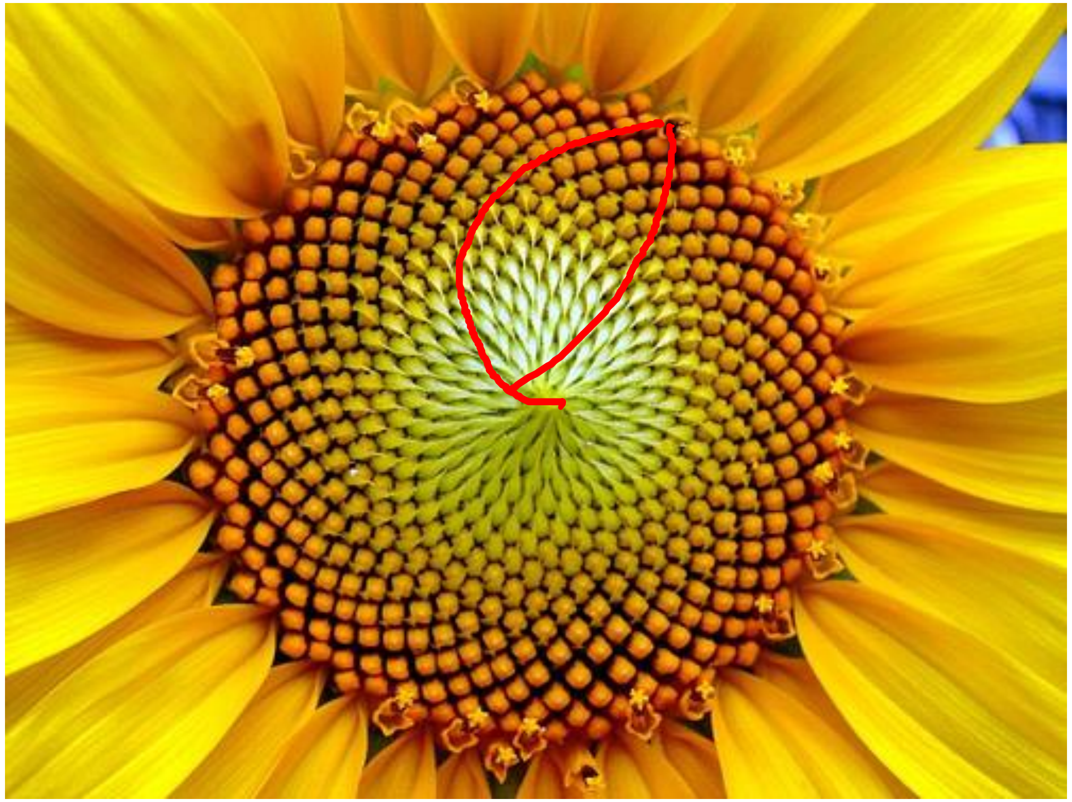
$a_4 = 2(-3) - 5 = -11$

$a_5 = 2(-11) - 5 = -27$

<u>Ex.</u> The Fibonacci sequence is defined as $a_0 = 1$, $a_1 = 1$, $a_k = a_{k-1} + a_{k-2}$. Write the first six terms.

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$





If n is a positive integer, <u>n factorial</u> is defined as

$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

As a special case, $0! = 1$.

Keep in mind that parentheses matter:

$2n! = 2 \cdot n! = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n)$

$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 2n$

$2 \cdot 5! = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)$

$(2 \cdot 5)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$

Ex. Write the first five terms of the sequence

$$a_n = \frac{2^n}{n!}$$

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{1 \cdot 2} = 2$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{1 \cdot 2 \cdot 3} = \frac{8}{6} = \frac{4}{3}$$

$$a_4 = \frac{2^4}{4!} = \frac{\cancel{2} \cdot \cancel{2 \cdot 2} \cdot 2}{1 \cdot \cancel{2} \cdot 3 \cdot \cancel{4}} = \frac{2}{3}$$

$$a_5 = \frac{2^5}{5!} = \frac{\cancel{2} \cdot \cancel{2 \cdot 2} \cdot 2 \cdot 2}{1 \cdot \cancel{2} \cdot 3 \cdot \cancel{4} \cdot 5} = \frac{4}{15}$$

$$2, 2, \frac{4}{3}, \frac{2}{3}, \frac{4}{15}, \ldots$$

Ex. Evaluate the factorial

a) $\dfrac{8!}{2!\cdot 6!} = \dfrac{\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot 7\cdot \cancel{8}^{4}}{1\cdot \cancel{2}\cdot \cancel{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}} = 28$

b) $\dfrac{\cancel{2!}\cdot \cancel{6!}^{6}}{\cancel{3!}_{3}\cdot \cancel{5!}} = 2$

c) $\dfrac{(n+1)!}{(n-1)!} = \dfrac{\cancel{1\cdot 2\cdot 3\cdot \ldots\cdot (n-1)}\cdot n\cdot (n+1)}{\cancel{1\cdot 2\cdot 3\cdot \ldots\cdot (n-1)}} = n(n+1)$

The Greek letter sigma (Σ) can be used to show the sum of many terms

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

i is called the <u>index</u> of the summation

n is the <u>upper limit</u> of the summation

1 is the <u>lower limit</u> of the summation

Ex. Find the sum

a) $\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$= 3 + 6 + 9 + 12 + 15 = \boxed{45}$

b) $\sum_{k=3}^{6} (1+k^2) = (1+3^2) + (1+4^2) + (1+5^2) + (1+6^2)$

$= 10 + 17 + 26 + 37 = \boxed{90}$

c) $\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$0! = 1$

The sum of all the terms of the infinite sequence is called an <u>infinite series</u>, and is denoted

$$\sum_{n=1}^{\infty} a_n = \underbrace{\underbrace{\underbrace{\underbrace{a_1}_{S_1} + a_2}_{S_2} + a_3}_{S_3} + \ldots + a_n}_{S_n} + \ldots$$

partial sums

Ex. Use the first 3 partial sums to evaluate the sum $\sum_{i=1}^{\infty} \frac{3}{10^i}$

$S_1 = \frac{3}{10^1} = .3$

$S_2 = \frac{3}{10^1} + \frac{3}{10^2} = .3 + .03 = .33$

$S_3 = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = .3 + .03 + .003 = .333$

$S_4 = .3333$

$S_\infty = .\overline{3} = \frac{1}{3}$