The Binomial Theorem

We are going to discuss how to expand something like $(x + y)^8$ without a ton of work

First, we need to define the "choose function"

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

We will also write ${}_nC_r$ for this function

Ex. Evaluate

a) ${}_8C_2 = \dfrac{\overset{4}{\cancel{8}}!\;\; 7\cdot\cancel{8}}{\underset{\cancel{2}}{2!}\,\cancel{6!}} = 28$

b) $\dbinom{10}{3} = \dfrac{\cancel{10!}\;\; 8\cdot 9\cdot 10}{\underset{1\cdot 2\cdot 3}{\cancel{3!}}\,\cancel{7!}} = \dfrac{\overset{4}{\cancel{8}}\cdot\overset{3}{\cancel{9}}\cdot 10}{1\cdot\cancel{2}\cdot\cancel{3}} = 120$

c) ${}_7C_0 = \dfrac{7!}{0!\,7!} = 1$

d) $\dbinom{8}{8} = \dfrac{8!}{8!\,0!} = 1$

Ex. Evaluate

a) ${}_7C_3 = \dfrac{\cancel{7!}^{\,5\cdot 6\cdot 7}}{3!\,\cancel{4!}} = \dfrac{5\cdot\cancel{6}\cdot 7}{1\cdot\cancel{2\cdot 3}} = 35$

b) $\dbinom{7}{4} = \dfrac{7!}{4!\,3!} = 35$

c) ${}_{12}C_1 = \dfrac{\cancel{12!}^{\,12}}{1!\,\cancel{11!}} = 12$

d) $\dbinom{12}{11} = 12$

When we expand $(x + y)^n$

- There will be $n + 1$ terms
- As the powers of x decrease, the powers of y increase
- The powers of x and y will add up to n in each term
- The coefficient of the term with x^r is ${}_nC_r$

Consider the expansion

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$${}_5C_3 = 10$$

An easy way to find these coefficients is Pascal's Triangle

1 ROW 0

1 1 ROW 1

1 2 1 ROW 2

1 3 3 1 ROW 3

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1 ROW 7

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Ex. Expand the binomial $(x + 1)^3$

$$1 \cdot x^3 + 3 \cdot x^2 \cdot 1^1 + 3 \cdot x^1 \cdot 1^2 + 1 \cdot 1^3$$

$$\boxed{x^3 + 3x^2 + 3x + 1}$$

$\underline{\text{Ex.}}$ Expand the binomial $(\underbrace{2x}\,\underbrace{-\,3})^4$

$1(2x)^4 + 4(2x)^3(-3)^1 + 6(2x)^2(-3)^2 + 4(2x)^1(-3)^3 + 1(-3)^4$

$2^4x^4 + 4(2)^3x^3(-3) + 6(2)^2x^2(-3)^2 + 4(2)x(-3)^3 + (-3)^4$

$16x^4 - 96x^3 + 216x^2 - 216x + 81$

Ex. Expand the binomial $(x^2 + 4)^3$

$$1(x^2)^3 + 3(x^2)^2(4)^1 + 3(x^2)^1(4)^2 + 1 \cdot 4^3$$

$$x^6 + 12x^4 + 48x^2 + 64$$

Ex. Find the coefficient of the term x^3 in the expansion of $(x + 2)^8$

$$\left({}_8C_3\right) x^3 (2)^5$$

$$56 x^3 (2)^5$$

$$1792 x^3$$

Ex. Find the coefficient of the term x^6y^5 in the expansion of $(3x - 2y)^{11}$

$$({}_{11}C_6)(3x)^6(-2y)^5$$

$$(462)\,3^6\,x^6\,(-2)^5\,y^5$$

$$\boxed{-10777536}\,x^6y^5$$