Math 200 – Linear Algebra

Andy Rosen

www.rosenmath.com

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Systems of Linear Equations

A system of linear equations (or linear system) refers to multiple equations involving multiple variables

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad\quad - 4x_3 = -7 \end{cases}$$

Linear means each variable is to the 1st power

Coefficients are the numbers multiplying the variables

In general, a linear equation would look like

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \longrightarrow 2(5) - 6.5 + 1.5(3) = 8 \\ x_1 \qquad\quad - 4x_3 = -7 \longrightarrow 5 \qquad - 4(3) = -7 \end{cases}$$

A solution is values of the variables that satisfy both equations.

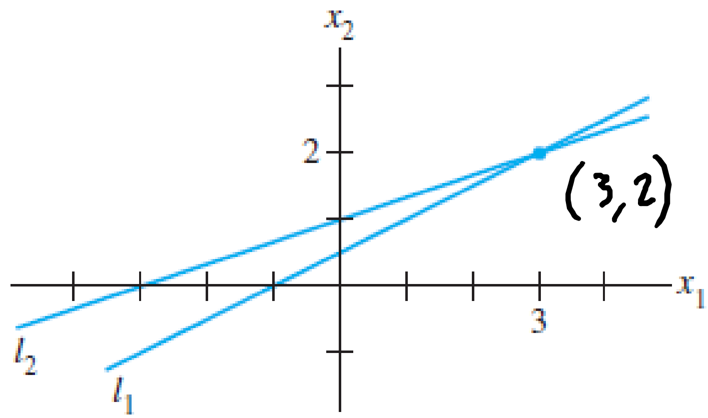
→ (5, 6.5, 3) is a solution to the system above

→ There are other solutions to this system. The collection of all possible solutions to a system is called the solution set

Two systems are equivalent if they have to same solution set.

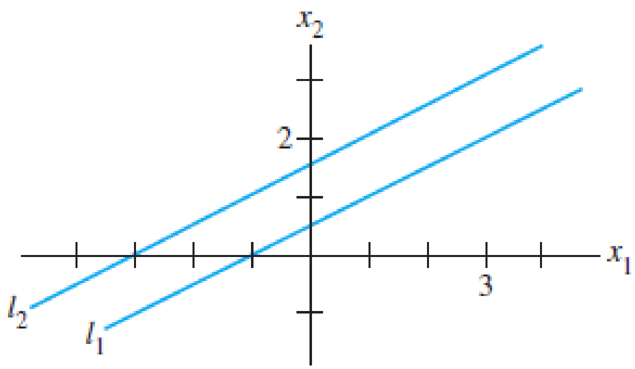
When solving a linear system with 2 variables and 2 equations, the solution is the point where the graphs of the lines intersect:

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$



$x_1 = 3$
$x_2 = 2$

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases}$$

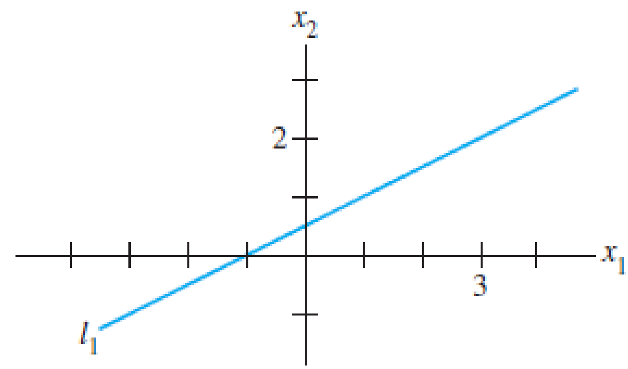


These lines are parallel

→ No solution

→ The system is inconsistent

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$$



These lines coincide

→ Infinitely many solution

A matrix is a rectangular array that can help us to streamline the solving of a system of equations

$$\begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & 9 \end{bmatrix}$$

rows columns

The size of this matrix is 2×3

By changing a system of equations into a matrix (<u>augmented matrix</u>), we can make it easier to work with

$$\begin{cases} x_1 - 4x_2 + 3x_3 = 5 \\ -x_1 + 3x_2 - x_3 = -3 \\ 2x_1 \qquad + 4x_3 = 6 \end{cases} \Rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right] \end{array}$$

The left side is called the <u>coefficient matrix</u>

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & 4 \end{bmatrix}$$

The following row operations will produce an equivalent system:

- Rows can be switched (<u>interchange</u>)

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Rightarrow} \begin{bmatrix} -1 & 3 & -1 & -3 \\ 1 & -4 & 3 & 5 \\ 2 & 0 & 4 & 6 \end{bmatrix}$$

The following row operations will produce an equivalent system :

- A row can be multiplied by a non-zero constant (<u>scaling</u>)

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{bmatrix} \underset{R_3 \to 4R_3}{\Rightarrow} \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 8 & 0 & 16 & 24 \end{bmatrix}$$

The following row operations will produce an equivalent system :

• A row can be replaced by the sum of itself and a multiple of another row (<u>replacement</u>)

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{bmatrix} \underset{R_3 \to R_2 + R_3}{\Rightarrow} \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 1 & 3 & 3 & 3 \end{bmatrix}$$

We are going to use row operations to put a matrix into echelon form

- Any row with all zeroes is at the bottom
- Each lead entry (first nonzero entry, reading left-to-right) has zeroes below it

If each lead entry is 1, and if each 1 has zeroes above and below it, we say the matrix is in reduced echelon form

These are in echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 3 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

This is also called <u>triangular form</u>.

These are in reduced echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

To put a matrix into reduced echelon form:

- Use scaling or interchange to place a non-zero number (preferably 1) in the upper-left entry, then use it with replacement to get zeroes below
- Move down and right, repeating the process to put the matrix in echelon form
- Starting with the bottom right 1, work backward to put the matrix in reduced echelon form

Ex. Solve the system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\Rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \end{array} \xRightarrow{R_3 \to 4R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xRightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \xRightarrow{R_3 \to 3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & \textcircled{1} & 3 \end{array}\right] \xRightarrow[R_2 \to 4R_3 + R_2]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & \textcircled{1} & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xRightarrow{R_1 \to 2R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{matrix} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{matrix}$$

Because there was a solution, the system is consistent.

A matrix in echelon form may be different depending on the row operations you choose

→ The reduced echelon form will be unique

Another option is to leave the augmented matrix in echelon form and then backsolve

Ex. Solve the system

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right] \xrightarrow{R_3 \to -5R_1 + 2R_3} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1 & 4 & -3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right] \longrightarrow 0 = 5$$

no solution

Because there was no solution, the system is inconsistent.

A <u>pivot position</u> in matrix *A* is a location in *A* that corresponds to a leading 1 in the reduced echelon form of *A*. A <u>pivot column</u> is a column of *A* that contains a pivot position.

→ The previous system was inconsistent because the rightmost column was a pivot column

Ex. Reduce the matrix to echelon form and locate the pivot columns

$$\begin{bmatrix} 0 & -3 & -6 & 4 & -9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} \textcircled{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & -9 \end{bmatrix}$$

$$\xrightarrow[R_3 \to 2R_1 + R_3]{R_2 \to R_1 + R_2} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & -9 \end{bmatrix} \xrightarrow[R_3 \to \frac{1}{5}R_3]{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & \textcircled{1} & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & -9 \end{bmatrix}$$

$$\xrightarrow[R_4 \to 3R_2 + R_4]{R_3 \to R_2 - R_3} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -18 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{} \begin{bmatrix} \textcircled{1} & 4 & 5 & -9 & -7 \\ 0 & \textcircled{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \textcircled{-5} & -18 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Col 1, 2, 4

Consider the system whose augmented matrix has been changed into the reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 \quad -5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{cases}$$

The variables x_1 and x_2 are called <u>basic variables</u>. The variable x_3 is called a <u>free variable</u> because it is free to be anything.

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 = x_3 \end{cases}$$

Choosing different values of x_3 results in different solutions

Ex. Find the general solution of the linear system whose augmented matrix has been reduced to

$$\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to 2R_3 + R_1} \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow{R_1 \to -2R_2 + R_1} \left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

Basic: x_1, x_3, x_5

Free: x_2, x_4

$$\Longrightarrow \begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 4x_4 &= 5 \\ x_5 &= 7 \end{aligned} \Longrightarrow \begin{aligned} x_1 &= -6x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= 4x_4 + 5 \\ x_4 &= x_4 \\ x_5 &= 7 \end{aligned}$$

This is called a parametric description of the solution set.

→ The free variables can be considered to be parameters

Note that the basic variables corresponded with the pivot columns.

If a system is consistent, then the solution set contains either

i. A unique solution, so there are no free variables, or
ii. Infinitely many solutions, so there is at least one free variable.