Vectors

A matrix with only one column is called a column vector, or simply a vector.

$\vec{u}$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The set of all vectors with 2 entries is $\mathbb{R}^2$ (read R-two), since each of the two entries can be any real number.

Two vectors are equal if the corresponding entries are equal.

Ordered pairs in the *xy*-plane, like vectors in $\mathbb{R}^2$, are represented by two numbers.

We can identify the plotted point (3,-1) with the column vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Sometimes, it is useful to include a directed line segment (arrow) from the origin to the point, though we are not interested in any of the points on the segment.

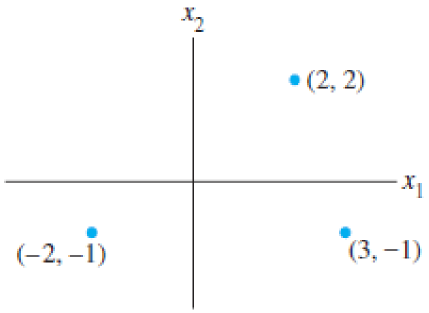


FIGURE 1 Vectors as points.

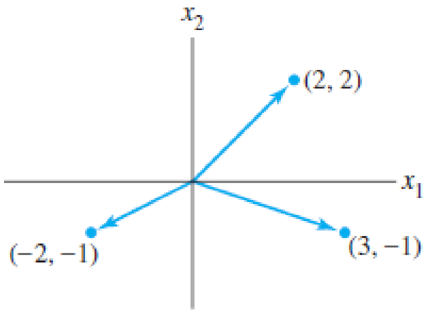


FIGURE 2 Vectors with arrows.

Adding and subtracting vectors means performing the operations on corresponding entries

Scalar multiplication means multiplying a vector by a constant (scalar)

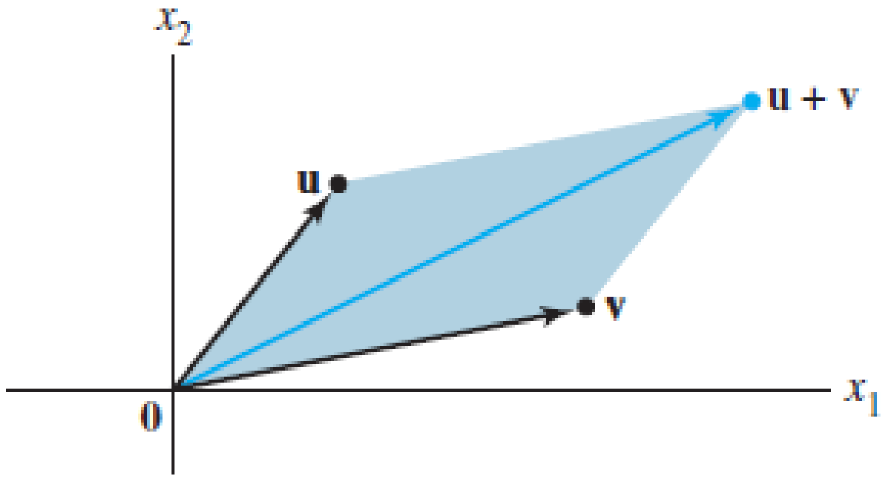
→ We do this by multiplying each entry by the constant

Ex. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

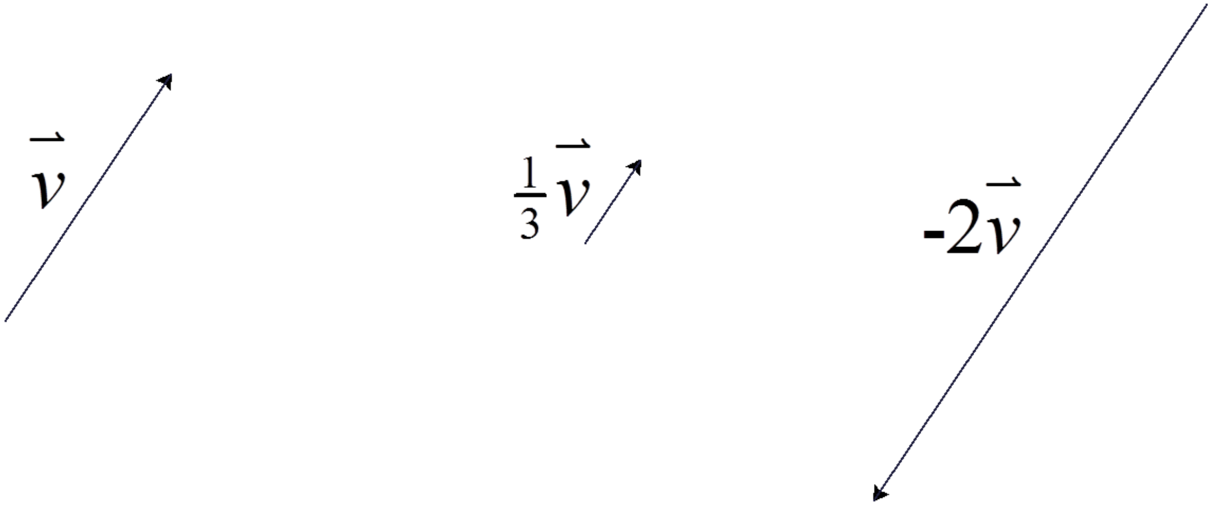
a. $3\mathbf{u} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$

b. $3\mathbf{u} - \mathbf{v} = \begin{bmatrix} 6 \\ -9 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$

If **u** and **v** in $\mathbb{R}^2$ are represented as points in the *xy*-plane, then **u** + **v** corresponds to the fourth vertex of the parallelogram formed by **u** and **v**.



Def. If *c* is a scalar and **v** is a vector, then *c***v** is the vector with the same direction as **v** that has length *c* times as long as **v**. If $c < 0$, then *c***v** goes in the opposite direction as **v**.

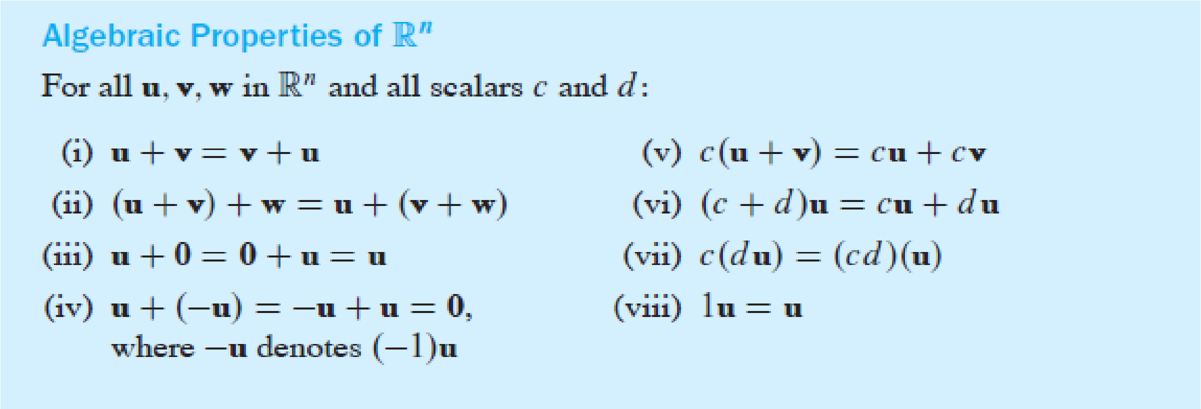


These ideas can be extended to n-dimensional space, $\mathbb{R}^n$.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$\vec{0}$

The <u>zero vector</u>, **0**, is the vector whose entries are all zero.



Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

A linear combination of vectors involves multiplying each vector by a constant coefficient and adding the results.

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

→ The vector $\mathbf{u} = \begin{bmatrix}14\\-7\end{bmatrix}$ is a linear combination of $\mathbf{v}_1 = \begin{bmatrix}2\\-3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}4\\1\end{bmatrix}$ because $\mathbf{u} = 3\mathbf{v}_1 + 2\mathbf{v}_2$.

The coefficients are called the weights of the combination

$$3\begin{bmatrix}2\\-3\end{bmatrix} + 2\begin{bmatrix}4\\1\end{bmatrix}$$

$$\begin{bmatrix}6\\-9\end{bmatrix} + \begin{bmatrix}8\\2\end{bmatrix}$$

$$\begin{bmatrix}14\\-7\end{bmatrix}$$

Ex. Determine if **b** can be written as a linear combination of **a**$_1$ and **a**$_2$.

$$\mathbf{a}_1 = \begin{bmatrix}1\\-2\\-5\end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix}2\\5\\6\end{bmatrix}, \mathbf{b} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 \overset{?}{=} \vec{b}$$

$$x_1 \begin{bmatrix}1\\-2\\-5\end{bmatrix} + x_2 \begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$\begin{bmatrix}x_1\\-2x_1\\-5x_1\end{bmatrix} + \begin{bmatrix}2x_2\\5x_2\\6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$\begin{bmatrix}x_1+2x_2\\-2x_1+5x_2\\-5x_1+6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$\begin{aligned} x_1+2x_2 &= 7\\ -2x_1+5x_2 &= 4\\ -5x_1+6x_2 &= -3 \end{aligned} \Rightarrow \left[\begin{array}{cc|c}1&2&7\\-2&5&4\\-5&6&-3\end{array}\right] \xrightarrow[R_3\to 5R_1+R_3]{R_2\to 2R_1+R_2} \left[\begin{array}{cc|c}1&2&7\\0&9&18\\0&16&32\end{array}\right]$$

$$\xrightarrow[R_3\to \frac{1}{16}R_3]{R_2\to \frac{1}{9}R_2} \left[\begin{array}{cc|c}1&2&7\\0&1&2\\0&1&2\end{array}\right] \xrightarrow[R_3\to R_2-R_3]{} \left[\begin{array}{cc|c}1&2&7\\0&1&2\\0&0&0\end{array}\right]$$

No pivot in right column

$\therefore$ System is consistent

$\therefore$ $\vec{b}$ is a lin. comb.

Notice that the columns of our augmented matrix were $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$.

→ We can abbreviate by writing $[\mathbf{a}_1 \quad \mathbf{a}_2 \mid \mathbf{b}]$

In general:

A vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \mid \mathbf{b}]$

Ex. Convert $\begin{cases} 3x_1 - 2x_2 + x_3 = 4 \\ -x_1 + 5x_2 + 2x_3 = 6 \\ 2x_1 - x_2 - 5x_3 = 2 \end{cases}$ to a vector equation.

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ -1 & 5 & 2 & 6 \\ 2 & -1 & -5 & 2 \end{array}\right]$$

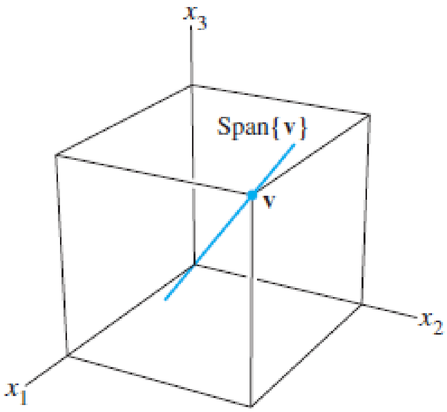
$$x_1\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

Def. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

That is, $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the set of all vectors that can be written $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$, where $c_1,\ldots,c_p$ are scalars.

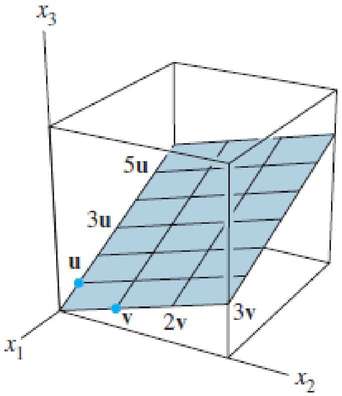
In $\mathbb{R}^3$:

Span{**v**} is the line through the origin and **v**:



$c\vec{v}$

Span{**u**,**v**} is the plane through the origin, **u** and **v**:



$c_1\vec{u} + c_2\vec{v}$

Ex. Determine if **b** is in the plane generated by Span{$\mathbf{a}_1, \mathbf{a}_2$}.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

$\Downarrow$

Is $\vec{b}$ a lin. comb. of $\vec{a}_1$ and $\vec{a}_2$?

$$\left[\begin{array}{cc|c} \enclose{circle}{1} & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array}\right] \xrightarrow[R_3 \to -3R_1 + R_3]{R_2 \to 2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & \enclose{circle}{-3} & 2 \\ 0 & -18 & 10 \end{array}\right] \xrightarrow[R_3 \to -6R_2 + R_3]{} \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & \enclose{circle}{-2} \end{array}\right]$$

Pivot in right col.

$\therefore$ Inconsist.

$\therefore$ $\vec{b}$ not in span$\{\vec{a}_1, \vec{a}_2\}$

The Matrix Equation

Let A be the matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_\mathrm{n}]$, where each of the **a**’s is a vector in $\mathbb{R}^m$, and let **x** be a vector in $\mathbb{R}^n$. Then the product $A\mathbf{x}$ is the linear combination of the columns of A using the entries of **x** as weights:

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Ex. $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Ex. $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

Ex. $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + 1\ ?$$

✗

Linear system:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & 1 \end{array}\right]$$

Vector equation:

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}2\\-5\end{bmatrix} + x_3\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$$

Matrix Equation:

$$\underset{A}{\begin{bmatrix}1 & 2 & -1\\0 & -5 & 3\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix}4\\1\end{bmatrix}}$$

$$A\vec{x} = \vec{b}$$

Linear systems can be expressed in 3 different ways, we can pick the one that's most convenient.

Ex. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, and b_3?

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right]$$

$$\xrightarrow[R_3 \to 3R_1 + R_3]{R_2 \to 4R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{array}\right] \xrightarrow[R_3 \to -2R_3 + R_2]{} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{array}\right]$$

It is possible that the pivot is in right col.

$\therefore$ not consist. for all b's

Thm. Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$. The following are equivalent (all are true or none are true):

i. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ in $\mathbb{R}^m$. $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots = \vec{b}$

ii. Every $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A

iii. The columns of A span $\mathbb{R}^m$ (every vector in $\mathbb{R}^m$ is in the span of the columns of A) → all lin. comb. of col. of A

iv. A has a pivot position in every row

Note: This is about the coefficient matrix, A, of a linear system, not the augmented matrix $[A \mid \mathbf{b}]$.

Ex. Can $A\mathbf{x} = \mathbf{b}$ be solved for any $\mathbf{b}$ in $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 0 & -1 & 6 \\ 7 & 1 & -1 & 14 \\ 5 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 6 \\ 7 & 1 & -1 & 14 \\ 5 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to -5R_1 + R_3]{R_2 \to -7R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 6 & -28 \\ 0 & 1 & 6 & -28 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_2 - R_3]{} \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 6 & -28 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no pivot in bottom row

$\therefore$ can't be solved for every $\vec{b}$

Ex. Do the columns of A span $\mathbb{R}^3$?

$$A = \begin{bmatrix} 7 & 1 & 2 \\ 5 & -1 & 6 \\ -2 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 1 & 2 \\ 5 & -1 & 6 \\ -2 & 0 & 4 \end{bmatrix} \xRightarrow[R_3 \to -\frac{1}{2}R_3]{} \begin{bmatrix} 7 & 1 & 2 \\ 5 & -1 & 6 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\xRightarrow{R_1 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & -2 \\ 5 & -1 & 6 \\ 7 & 1 & 2 \end{bmatrix} \xRightarrow[R_3 \to -7R_1 + R_3]{R_2 \to -5R_1 + R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 16 \\ 0 & 1 & 16 \end{bmatrix} \xRightarrow[R_3 \to R_2 + R_3]{} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 16 \\ 0 & 0 & 32 \end{bmatrix}$$

Pivot in each row

$\therefore$ columns span $\mathbb{R}^3$

The <u>identity matrix</u> is a square matrix that has ones on its main diagonal and zeroes as every other entry

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying any vector by I results in the same vector

$$I\mathbf{x} = \mathbf{x}$$