Solution Sets of Linear Systems

The linear system $A\mathbf{x} = \mathbf{0}$ is called homogeneous.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

This system always has at least 1 solution, where all the x's are 0. This is called the trivial solution.

Thm. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

→ So the homogeneous system has either one trivial solution or infinitely many solutions.

<u>Ex.</u> Describe the solution set of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to R_1 + R_2} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to 3R_2 + R_3]{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to -5R_2 + R_1} \left[\begin{array}{ccc|c} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\Longrightarrow \begin{array}{l} x_1 - \frac{4}{3}x_3 = 0 \\ x_2 = 0 \end{array} \Longrightarrow \begin{array}{l} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \\ x_3 = x_3 \end{array} \Longrightarrow \vec{x} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} x_3$$

$$\text{span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

1 free variable resulted in a line in $\mathbb{R}^3$.

Ex. Describe the solution set of $10x_1 - 3x_2 - 2x_3 = 0$

$$\left[\begin{array}{ccc|c} 10 & -3 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{-3}{10} & \frac{-1}{5} & 0 \end{array}\right] \Rightarrow x_1 - \frac{3}{10}x_2 - \frac{1}{5}x_3 = 0$$

$$\Rightarrow \begin{aligned} x_1 &= \tfrac{3}{10}x_2 + \tfrac{1}{5}x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} \frac{3}{10}x_2 + \frac{1}{5}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{5}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix} x_3$$

$$\text{span}\left\{ \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2 free variables resulted in a plane in $\mathbb{R}^3$.

If A has no free variables:

- Trivial solution
- The point $\mathbf{0}$ in $\mathbb{R}^3$

If A has 1 free variable:

- A line in $\mathbb{R}^3$ that passes through the origin
- Can be described parametrically by $\mathbf{x} = t\mathbf{v}_1$.

If A has 2 free variables:

- A plane in $\mathbb{R}^3$ that passes through the origin
- Can be described parametrically by $\mathbf{x} = s\mathbf{v}_1 + t\mathbf{v}_2$.

→ Note this represents Span$\{\mathbf{v}_1, \mathbf{v}_2\}$

When we write our solution sets in this form, it is called the parametric vector form.

If $\mathbf{b} \neq \mathbf{0}$, the linear system $A\mathbf{x} = \mathbf{b}$ is called non-homogeneous.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Ex. Describe the solution set of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{aligned} x_1 - \tfrac{4}{3}x_3 &= -1 \\ x_2 &= 2 \end{aligned} \Longrightarrow \begin{aligned} x_1 &= \tfrac{4}{3}x_3 - 1 \\ x_2 &= 2 \\ x_3 &= x_3 \end{aligned}$$

$$\Longrightarrow \vec{x} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ has no solutions if:

- $A\mathbf{x} = \mathbf{b}$ is inconsistent

$A\mathbf{x} = \mathbf{b}$ has 1 solution if:

- The corresponding homogeneous system had only the trivial solution

$A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if:

- The corresponding homogeneous system had infinitely many solutions
- Solutions would be 1 vector plus a linear combination of vectors that satisfy the corresponding homogeneous system.
- $\mathbf{x} = \mathbf{p} + t\mathbf{v}_1$ → a line not through the origin
- $\mathbf{x} = \mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2$ → a plane not through the origin

Prove the previous result:

Assume $\vec{p}$ is solution to $A\vec{x} = \vec{b}$. $\longrightarrow$ $A\vec{p} = \vec{b}$

Assume $t\vec{v}$ is solution to $A\vec{x} = \vec{0}$. $\longrightarrow$ $A(t\vec{v}) = \vec{0}$

Show $\vec{p} + t\vec{v}$ is solution $A\vec{x} = \vec{b}$.

$$A(\vec{p} + t\vec{v}) \stackrel{?}{=} \vec{b}$$

$$A\vec{p} + A(t\vec{v}) \stackrel{?}{=} \vec{b}$$

$$\vec{b} + \vec{0} = \vec{b} \quad \checkmark$$

Linear Independence

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is linearly dependent if there exist constants $x_1, x_2, \ldots, x_p$ (not all zero) such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

→ This equation is called a linear dependence relation.

→ The set is linearly independent if $x_1 = x_2 = \ldots = x_p = 0$ is the only solution.

Ex. Determine if the vectors are dependent. Find a linear dependence relation. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \to -3R_1 + R_3]{R_2 \to -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \xrightarrow[R_3 \to -\frac{1}{6}R_3]{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_2 - R_3]{} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

no pivot in 3^{rd} col

$\therefore$ many solns.

$\therefore$ vectors lin. dep.

$$\xrightarrow{R_1 \to -4R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \Longrightarrow \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned} \xrightarrow{x_3 = 1} \begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 1 \end{aligned}$$

$$\boxed{2v_1 - v_2 + v_3 = \vec{0}} \leftarrow \text{dep. relation}$$

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

→ Note this is the same as our homogeneous equation $A\mathbf{x} = \mathbf{0}$, where the vectors are the columns of A.

Thm. The following are equivalent:

i. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

ii. The columns of A are linearly independent

iii. The linear system with augmented matrix $[A \mid \mathbf{0}]$ has no free variables

iv. A has a pivot in each column

Ex. Determine if the vectors are dependent.

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \xRightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 5 & 8 & 0 \end{bmatrix} \xRightarrow[R_3 \to -5R_1 + R_3]{} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix} \xRightarrow[R_3 \to 2R_2 + R_3]{} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix}$$

pivot in every col.

$\therefore$ vectors are lin. indep.

Ex. Determine if the vectors are dependent.

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \xRightarrow[R_2 \to -3R_2 + R_1]{} \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}$$

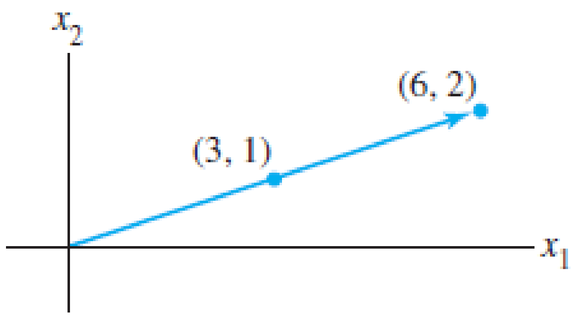
no pivot in every col.
∴ lin. dep.

b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

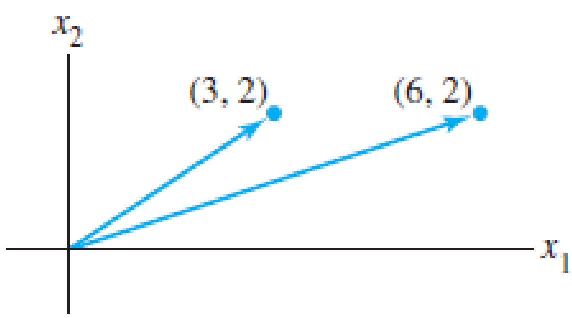
$$\begin{bmatrix} 3 & 6 \\ 2 & 2 \end{bmatrix} \xRightarrow[R_2 \to \frac{1}{2} R_2]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 \\ 3 & 6 \end{bmatrix} \xRightarrow[R_2 \to -3R_1 + R_2]{} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

pivot in every col.
∴ lin. indep.

Two vectors are linearly dependent if one is a multiple of the other.



Linearly dependent



Linearly independent

Note: This doesn’t work for more than 2 vectors!

Thm. A set of two or more vectors is linearly dependent if and only if at least one is a linear combination of the others.

Ex. Describe the set spanned by **u** and **v**. Explain why a vector **w** is in Span{**u**,**v**} if and only if {**u**, **v**, **w**} is linearly dependent.

$$\cancel{\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}}$$

$$\text{span}\{\vec{u}, \vec{v}\} = \{w \mid \vec{w} = a\vec{u} + b\vec{v}\}$$

Assume $\vec{u}, \vec{v}, \vec{w}$ are dep.

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{0} \quad (x_1, x_2, x_3 \text{ not all } 0)$$

$$\Longrightarrow \vec{w} = \frac{-x_1}{x_3}\vec{u} - \frac{x_2}{x_3}\vec{v}$$

$$\Longrightarrow \vec{w} = a\vec{u} + b\vec{v} \Longrightarrow \vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$$

Thm. If a set contains more vectors than there are entries in each vector, then the set is dependent.

Ex. Show that the set is dependent. $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

$$\xcancel{\begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 2 \end{bmatrix}}$$

Too many vectors

$\therefore$ lin. dep.

Thm. If a set contains the zero vector, then the set is dependent.

$\vec{u}, \vec{v}, \vec{w}, \vec{0}$

$$x_1 \vec{u} + x_2 \vec{v} + x_3 \vec{w} + x_4 \vec{0} = \vec{0}$$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 12$

Ex. Determine if the set is dependent.

a. $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \right\}$ too many vectors

b. $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\}$

zero vector

c. $\left\{ \overset{\vec{u}}{\begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}}, \overset{\vec{v}}{\begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}} \right\}$

$\vec{v} = -3\vec{u}$