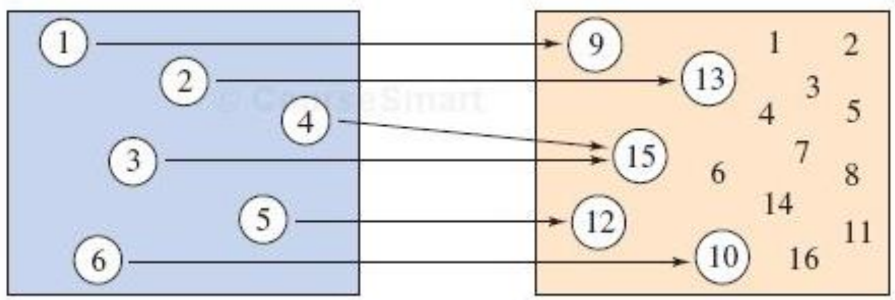
Intro to Linear Transformations

Def. A function f from set A to set B is a relation that assigns to each element x in set A exactly one element y in set B.



Set A | Set B

Domain | Codomain

$$\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

We can think of A as transforming $\mathbf{x}$ in $\mathbb{R}^4$ to $\mathbf{b}$ in $\mathbb{R}^2$.

A <u>transformation</u> (or <u>function</u> or <u>mapping</u>) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

$T: \mathbb{R}^n \to \mathbb{R}^m$

$\mathbb{R}^n$ is the domain

$\mathbb{R}^m$ is the codomain

The set of all $T(\mathbf{x})$ is called the <u>range</u>

→ The range is a subset of the codomain

The rest of this section will focus on mappings associated with matrix multiplication

$$\mathbf{x} \mapsto A\mathbf{x}$$

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

a. If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $T(\mathbf{u})$.

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ -7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

$$\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

b. If $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, find an $\mathbf{x}$ whose image under T is $\mathbf{b}$.

$$\vec{x} \longrightarrow A\vec{x} = \vec{b}$$

$$A\vec{x} = \vec{b} \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 1 & -1/2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \frac{3}{2} \\ x_2 = -\frac{1}{2} \end{array}$$

$$\Downarrow$$

$$\vec{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

is one-to-one

Was this answer unique?

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

c. If $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, find an $\mathbf{x}$ whose image under T is $\mathbf{c}$.

$A\vec{x} = \vec{c}$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 5/2 \end{array}\right]$$

no solution

not onto

$$?? \longrightarrow \vec{c}$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

<u>Ex.</u> Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by
$T(\mathbf{x}) = A\mathbf{x}$.

d. Find all $\mathbf{x}$ that are mapped into the zero vector.

$A\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 3 & 5 & 0 \\ -1 & 7 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 14 & 0 \\ 0 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

Ex. Find the image of $\mathbf{x}$ under the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$$

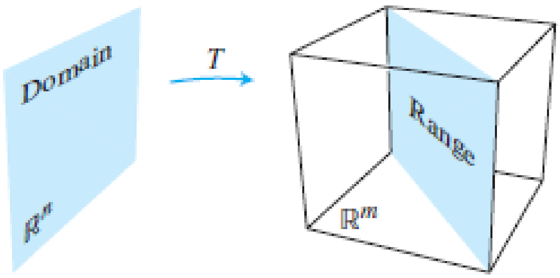
$$\begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix} \to \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$$

This projects the point onto the x_1x_2-plane.

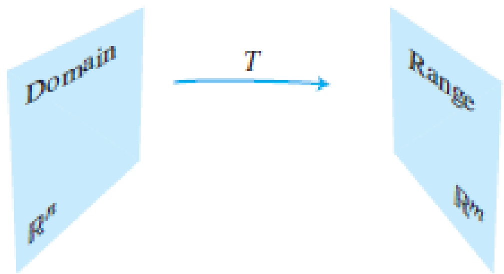
A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>onto</u> $\mathbb{R}^m$ if every $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least* one $\mathbf{x}$ in $\mathbb{R}^n$.

→ The range makes up the entire codomain

→ Every vector in $\mathbb{R}^m$ is the output at least once



T is *not* onto $\mathbb{R}^m$



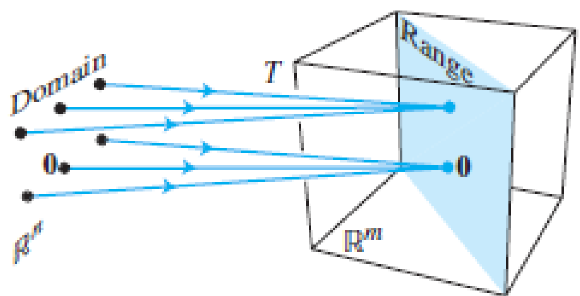
T *is* onto $\mathbb{R}^m$

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>one-to-one</u> if every **b** in $\mathbb{R}^m$ is the image of *at most* one **x** in $\mathbb{R}^n$.

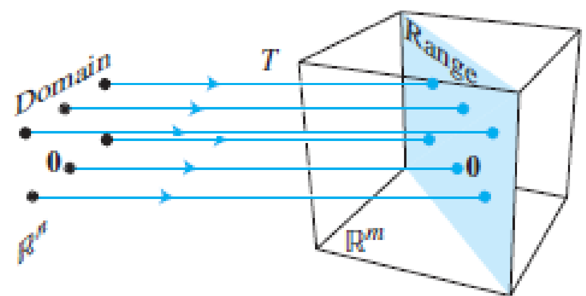
→ Every vector in the range is an output exactly once

→ Not all vectors in $\mathbb{R}^m$ are outputs

→ $T(\mathbf{x})$ has either a unique solution or no solution



T is *not* one-to-one



T *is* one-to-one

Ex. Define T: $\mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T one-to-one?

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

onto: Every $\vec{b}$ is an output?

$A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$?

$\left[\begin{array}{cccc|c} 1 & -4 & 8 & 1 & \\ 0 & 2 & -1 & 3 & \vec{b} \\ 0 & 0 & 0 & 5 & \end{array}\right]$ will be consist?

yes, it is onto because A has pivot in every row

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one-to-one: Solutions to $A\vec{x} = \vec{b}$ are unique?

$\left[\begin{array}{cccc|c} 1 & -4 & 8 & 1 & \\ 0 & 2 & -1 & 3 & \vec{b} \\ 0 & 0 & 0 & 5 & \end{array}\right]$ has unique solution?

no, not one-to-one because not pivot in every column

We remember properties of vector/matrix/scalar addition and multiplication:

Distributive: $A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Commutative: $A(c\mathbf{u}) = cA(\mathbf{u})$

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

These lead to the properties of a <u>linear</u> transformation T.

For any linear transformation,

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

In particular, $T(\mathbf{0}) = \mathbf{0}$.

→ This can be generalized to be true for any number of vectors. This is called the superposition principle.

Ex. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = 3\mathbf{x}$. Show that T is a linear transformation.

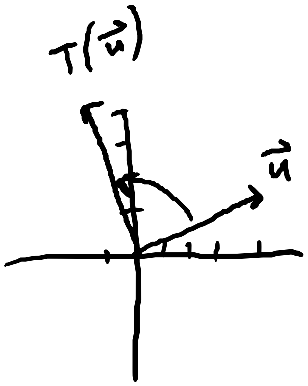
$$T(\vec{u}+\vec{v}) = 3(\vec{u}+\vec{v}) = 3\vec{u} + 3\vec{v} = T(\vec{u}) + T(\vec{v}) \checkmark$$

$$T(c\vec{u}) = 3(c\vec{u}) = 3c\vec{u} = c3\vec{u} = c(3\vec{u}) = cT(\vec{u}) \checkmark$$

What does this transformation represent graphically?

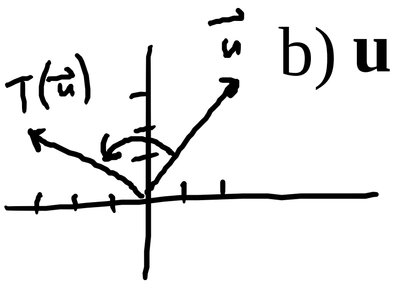
Ex. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\mathbf{x}$.

Find $T(\mathbf{u})$:



a) $\mathbf{u} = \begin{bmatrix}4\\1\end{bmatrix}$

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} = 4\begin{bmatrix}0\\1\end{bmatrix} + 1\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0\\4\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix}$$



b) $\mathbf{u} = \begin{bmatrix}2\\3\end{bmatrix}$

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = 2\begin{bmatrix}0\\1\end{bmatrix} + 3\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}-3\\0\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix}$$

rotate $90°$ CCW

What does this transformation represent graphically?

Matrix of a Linear Transformation

We have been talking about different linear transformations, not just ones that are matrix multiplication.

In fact, all linear transformations can be represented by a matrix multiplication.

To find the matrix, we will be using the columns of I_n, which we will call $\mathbf{e}_1$, $\mathbf{e}_2$, etc.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These are called the standard basis vectors of $\mathbb{R}_3$.

Ex. Suppose T is a linear transformation such that $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. Describe the image of an arbitrary $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right) = T\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1\begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thm. If $T{:}\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, there is a unique $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$.

→ The columns of A will be the transformation of the columns of I. In other words:

$$A = [T(\mathbf{e}_1) \quad \dots \quad T(\mathbf{e}_n)]$$

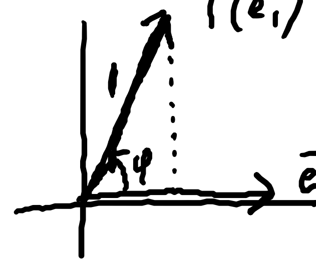
→ This is called the standard matrix for the linear transformation.

→ Please note mapping $\mathbb{R}^n \to \mathbb{R}^m$ requires a matrix that is $m \times n$.

Ex. Find the standard matrix for the transformation that rotates each point in $\mathbb{R}^2$ counterclockwise about the origin through an angle φ.

$$T(\vec{e}_1) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$$

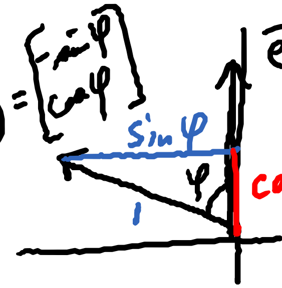
$$T(\vec{e}_1) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$$



$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$



$$T(\vec{e}_2) = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}$$

p. 73-75 has the standard matrices for several common geometric linear transformations.

→ Even more transformations come from the composition of transformations.

Ex. Define $T: \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T one-to-one?

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

It <u>is</u> onto because pivot in each <u>row</u>.

It <u>is not</u> one-to-one because not pivot in each <u>column</u>.

Thm. Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. The following are equivalent:

i. T is one-to-one.

ii. A has a pivot in each column.

iii. A has no free variables.

iv. The columns of A are linearly independent.

v. The equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

→ This links us with all of the equivalent statements from last class.

Thm. Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. The following are equivalent:

i. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$. $\Rightarrow$ $A\vec{x} = \vec{b}$ is consist. for every $\vec{b}$

ii. A has a pivot in each row.

iii. The columns of A span $\mathbb{R}^m$.

Ex. Let $T(x_1,x_2) = (3x_1 + x_2,\ 5x_1 + 7x_2,\ x_1 + 3x_2)$.
Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? Is T one-to-one?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

Is it onto? Is there a pivot in each row? no

Is it one-to-one? Is there a pivot in each column? yes

columns not multiples.
$\therefore$ columns are indep.
$\therefore$ one-to-one

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1,x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (0,-1,4)$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{bmatrix} \Longrightarrow T(\vec{x}) = \underbrace{\begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix}}_{A} \vec{x} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & 4 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 2 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -8 \end{array}\right]$$

no solution