Matrix Operations

$$\begin{bmatrix} 3 & 5 & -2 & 3 \\ 1 & 0 & 9 & 7 \\ -4 & 8 & 6 & -3 \end{bmatrix}$$

The 5 is entry a_{12} because it is in the 1^{st} row and 2^{nd} column

Entries a_{11}, a_{22}, etc. are called the main diagonal

A diagonal matrix is a square ($n \times n$) matrix whose nondiagonal entries are 0.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Two matrices are equal if they have the same order and if the corresponding entries are equal

Adding and subtracting matrices means performing the operations on corresponding entries

- The matrices must have the same order, and the result will also have that order

Ex.

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

$\underset{2\times 2}{}\quad \underset{2\times 2}{}\quad \underset{2\times 2}{}$

2x2 2x2 2x2

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 1 & 4 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 \\ 1 & 0 & 8 \end{bmatrix}$

2x3 2x3 2x3

Scalar multiplication means multiplying a matrix by a constant

- We do this by multiplying each entry by the constant

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

<u>Ex.</u> Let $A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

a. $3A = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

b. $3A - B$

When multiplying two matrices, we take a row from the first matrix and multiply it by a column from the second matrix

The orders have to match up:

$$\underset{4\times 3}{A} \times \underset{3\times 7}{B} = \underset{4\times 7}{AB}$$

Ex. $\underset{2\times \underline{3}}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \times \underset{\underline{3}\times 2}{\begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix}}$

$$(1)(-2)+(0)(1)+(3)(-1)=-5$$

$$(1)(4)+(0)(0)+(3)(1)=7$$

$$(2)(-2)+(-1)(1)+(-2)(-1)=-3$$

$$(2)(4)+(-1)(0)+(-2)(1)=6$$

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -3 & 1 & 6 \\ 4 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 17 & 12 \\ 7 & 35 & 30 \end{bmatrix}$

$2 \times \underline{2}$ $\quad \underline{2} \times 3 \quad$ 2×3

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$
for any scalar r

e. $\underset{m \times n}{I_m A} = \underset{m \times n}{A} = AI_n$ (identity for matrix multiplication)

$\underset{2\times 3}{A}\ \underset{3\times 2}{B} = 2 \times 2$	$\underset{3\times \underline{2}}{B}\ \underset{\underline{2}\times 3}{A} = 3 \times 3$
$\underset{2\times 2}{A}\ \underset{2\times 3}{B} = 2 \times 3$	$\underset{2\times \underline{3}}{B}\ \underset{\underline{2}\times 2}{A} = ?\ ?$

$$\underline{\text{Ex.}} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 7 & 35 \end{bmatrix}$$

$$\underset{2\times 2}{} \qquad \underset{2\times 2}{} \qquad \underset{2\times 2}{}$$

Ex. $\begin{bmatrix} -3 & 1 \\ 4 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 28 & 40 \end{bmatrix}$

$2 \times 2 \quad\quad 2 \times 2 \quad\quad 2 \times 2$

Cautions

i. In general, $AB \neq BA$. If fact, depending on the sizes, both products may not be possible.

ii. Cancellation laws do not hold. In other words, if $AB = AC$, it may not be true that $B = C$.

iii. If $AB = 0$, it may not be true that $A = 0$ or $B = 0$.

→ A^T (transpose) switches a_{ij} with a_{ji}

Ex. Find the transpose of each matrix:

a. $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 0 & -6 \\ -1 & 4 & 10 \\ 3 & 6 & -5 \end{bmatrix}$

b. $B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$

$B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$

c. $C = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & -2 \end{bmatrix}$

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

$$\big(\underset{4\times 3}{A}\ \underset{3\times 7}{B}\big)^T = \underset{7\times 4}{}$$

$$\cancel{\underset{3\times 4}{A^T}\ \underset{7\times 3}{B^T}}$$

$$\underset{7\times \underline{3}}{B^T}\ \underset{\underline{3}\times 4}{A^T} = \underset{7\times 4}{}$$

Inverse Matrices

2 and ½ are multiplicative inverses because

$$2 \times \frac{1}{2} = 1 \rightarrow 2^{-1}$$

The inverse of matrix A is written A^{-1}, and

$$AA^{-1} = I \text{ and } A^{-1}A = I$$

where I is the identity matrix

If a matrix has an inverse, we say that it is invertible

- Otherwise, we say that it is singular
- Only square matrices can be invertible

Ex. Show that $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ are inverses

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

For a 2×2 matrix, there's a quick way to find the inverse:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The quantity $ad - bc$ is called the determinant of the matrix:

$$\det A = ad - bc$$

We'll come back to this in the future…

Ex. Find the inverse

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{6-2}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

b. $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

$$B^{-1} = \frac{1}{6-6}\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = ??$$

singular

To solve the equation $ax = b$, we multiply by the multiplicative inverse $\frac{1}{a}$:

$$ax = b$$

$$\frac{1}{a}ax = \frac{1}{a}b$$

$$x = \frac{b}{a}$$

To solve a matrix equation, we do the same

$$A\mathbf{x} = \mathbf{b}$$

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Ex. Solve the system $\begin{cases} 3x_1 + 4x_2 = -2 \\ 5x_1 + 3x_2 = 4 \end{cases}$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 3 \end{bmatrix}}_{A} \vec{x} = \underbrace{\begin{bmatrix} -2 \\ 4 \end{bmatrix}}_{\vec{b}}$$

$$\boxed{\begin{aligned} A^{-1} &= \frac{1}{9-20}\begin{bmatrix} 3 & -4 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{-11}\begin{bmatrix} 3 & -4 \\ -5 & 3 \end{bmatrix} \end{aligned}}$$

$$\begin{aligned} \vec{x} &= A^{-1}\vec{b} \\ &= \frac{1}{-11}\begin{bmatrix} 3 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ &= \frac{1}{-11}\begin{bmatrix} -22 \\ 22 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -2 \end{bmatrix} \end{aligned}$$

Thm.

$= B$

i. $(A^{-1})^{-1} = A$

ii. $(AB)^{-1} = B^{-1}A^{-1}$

iii. $C =$ $(A^T)^{-1} = (A^{-1})^T$

$D =$ Let's prove these results.

$A \cdot B = A(A^{-1}) = I$

$(B^{-1}A^{-1})C = B^{-1}\underbrace{A^{-1}A}B = B^{-1}B = I$

$(A^{-1})^T D = (A^{-1})^T A^T = (AA^{-1})^T$

$= I^T = I$

For larger matrices, to find an inverse matrix we use row operations

- Create the matrix $[A \mid I]$
- Perform row operations to make the left side into I
- The result will be $[I \mid A^{-1}]$

Ex. Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

Thm. Invertible Matrix Theorem

Let A be $n \times n$. The following are equivalent:

i. A is invertible

ii. A is row equivalent to I.

iii. A has n pivot positions (one in each row and column).

iv. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

v. The columns of A are linearly independent.

vi. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

vii. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.

viii. The columns of A span $\mathbb{R}^n$.

ix. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

A is invertible

$\Updownarrow$

A is row equiv. to I

$\Updownarrow$

A has n pivot points

⤢

Pivot in every row

$\Updownarrow$

$A\mathbf{x} = \mathbf{b}$ has at least one solution

$\Updownarrow$

Columns span $\mathbb{R}^n$

$\Updownarrow$

Transformation is onto

⤡

Pivot in every column

$\Updownarrow$

$A\mathbf{x} = \mathbf{b}$ has at most one solution

$\Updownarrow$

Columns are linearly indep.

$\Updownarrow$

$A\mathbf{x} = \mathbf{0}$ only trivial solution

$\Updownarrow$

Transformation is one-to-one

Ex. Determine if A is invertible.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

pivot in every row and column

$\therefore$ A is inv.

Matrices A and B are inverses if $AB = I$ and $BA = I$.

→ Transformations T and S are inverses if

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad S(T(\mathbf{x})) = \mathbf{x}$$

In fact, if A is the standard matrix for T, then A^{-1} is the standard matrix for T^{-1}.