Properties of Determinants

Ex. Find the determinant

a. $\begin{vmatrix} 1 & 2 \\ 2 & -6 \end{vmatrix}$

$-6 - 4$

-10

b. $\begin{vmatrix} 2 & -6 \\ 1 & 2 \end{vmatrix}$

$4 - -6$

10

If two rows are interchanged, the determinant changes signs.

Ex. Find the determinant

a. $\begin{vmatrix} 1 & 2 \\ 2 & -6 \end{vmatrix}$

$-6-4$

-10

b. $\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix}$

$-3-2$

-5

If a row is multiplied by a scalar, the determinant is multiplied by the scalar (factor out of row).

Ex. Find the determinant

a. $\begin{vmatrix} 1 & 2 \\ 2 & -6 \end{vmatrix}$ $\xrightarrow{R_2 \to -2R_1 + R_2}$ b. $\begin{vmatrix} 1 & 2 \\ 0 & -10 \end{vmatrix}$

$-6-4$

-10

$-10-0$

-10

If a row is replaced by its sum with a multiple of another row, the determinant doesn't change.

Ex. Find the determinant

a. $\begin{vmatrix} 1 & 3 \\ 2 & -6 \end{vmatrix}$

$-6-6$

-12

b. $\begin{vmatrix} 1 & 2 \\ 3 & -6 \end{vmatrix}$

$-6-6$

-12

$\det A^{T} = \det A$

[These properties also work when doing column operations.]

We can make determinants easer to evaluate by using row operations (especially 4x4).

Ex. Find the determinant of $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

$$\begin{vmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=\!=} -\begin{vmatrix} 1 & -1 & 0 \\ -2 & 2 & 3 \\ 0 & 1 & 4 \end{vmatrix} \overset{R_2 \to R_2 + 2R_1}{=\!=} -\begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 4 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=\!=} \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{vmatrix} = (1)(1)(3) = \boxed{3}$$

Ex. Find the determinant of $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2(1)(3)(-6)(1)$$

$R_3 \to R_3 + 4R_2$ $\qquad$ $R_4 \to R_4 + \frac{-1}{2}R_3$

$$= \boxed{-36}$$

Ex. Find the determinant of $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 3 & 4 \end{bmatrix}$

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\vdots$$

$$= 2\begin{vmatrix} 1 & - & - & - \\ 0 & 3 & - & - \\ 0 & 0 & -6 & - \\ 0 & 0 & 0 & 0 \end{vmatrix} = 2(1)(3)(-6)(0)$$

$$= 0$$

A square matrix is invertible (and everything that goes with that) iff the determinant is non-zero.

Thm. Invertible Matrix Theorem

Let A be $n \times n$. The following are equivalent:

i. A is invertible

ii. A is row equivalent to I.

iii. A has n pivot positions (one in each row and column).

iv. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

v. The columns of A are linearly independent.

vi. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

vii. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.

viii. The columns of A span $\mathbb{R}^n$.

ix. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

x. The determinant of A is not zero

Ex. Verify that det $(AB) = (\det A)(\det B)$

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$\det A = 12 - 3 = 9$ $\det B = 8 - 3 = 5$

$$AB = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix} \rightarrow \det(AB) = 325 - 280 = 45$$

$$(\det A)(\det B) = \det(AB)$$

Caution: $\det(A + B) \neq \det A + \det B$

Ex. Compute det (B^3)

$$B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\det(B^3) = \det(B \cdot B \cdot B)$$
$$= (\det B)(\det B)(\det B)$$
$$= (\det B)^3$$
$$= (8-3)^3$$
$$= 5^3 = 125$$

Ex. Evaluate $\det\left(\begin{bmatrix} 7 & 8 & 1 & 0 \\ 0 & 5 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 \\ 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right) = (-35)(-48)$

-35

$3(-1)^{2+2}\begin{vmatrix} 4 & 0 & 0 \\ 1 & 2 & 6 \\ 0 & 1 & 1 \end{vmatrix}$

$3(4)(-1)^{1+1}\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix}$

$12(2-6)$

-48

Applications of Determinants

It is possible to solve a system of equations by finding a bunch of determinants:

Cramer's Rule

Consider the problem of solving $A\mathbf{x} = \mathbf{b}$. Let $A_1(\mathbf{b})$ be the matrix obtained from A by replacing column 1 with $\mathbf{b}$. Then

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A}$$

This process can be repeated to solve for the other variables.

Ex. Solve $\begin{cases} 4x_1 - 2x_2 = 10 \\ 3x_1 - 5x_2 = 11 \end{cases}$

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\det A = -20 - {}^{-}6 = -14$$

$$A_1(\vec{b}) = \begin{bmatrix} 10 & -2 \\ 11 & -5 \end{bmatrix} \quad \det A_1(\vec{b}) = -50 - {}^{-}22 = -28 \rightarrow x_1 = \frac{-28}{-14} = 2$$

$$A_2(\vec{b}) = \begin{bmatrix} 4 & 10 \\ 3 & 11 \end{bmatrix} \quad \det A_2(\vec{b}) = 44 - 30 = 14 \longrightarrow x_2 = \frac{14}{-14} = -1$$

Generally, it's quicker to do row reduction.

Thm. Let A be $n \times n$ and let C_{ij} be the cofactor for entry a_{ij}. Then

$$A^{-1} = \frac{1}{\det A} C^{\mathrm{T}}$$

C^{T} is called the adjugate (or classical adjoint) or A, and can be denoted adj A.

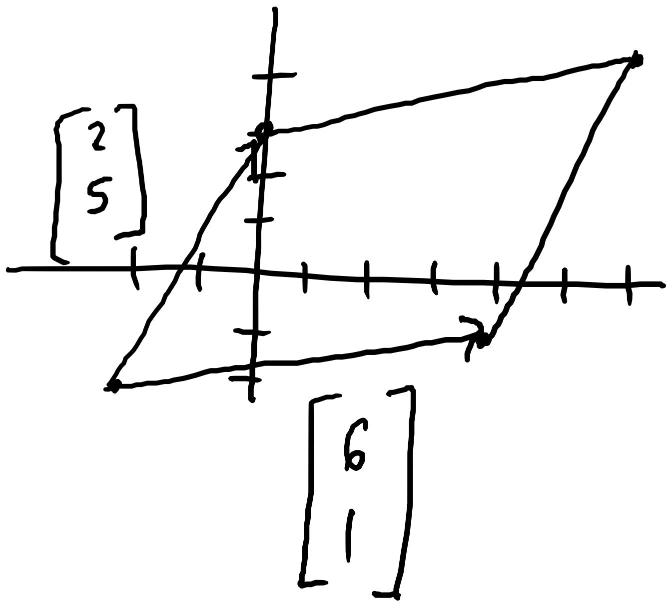
Generally, it's quicker to use the other method for finding A^{-1}.

Ex. Let $A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$, find A^{-1}.

Thm. If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Ex. Find the area of the parallelogram with vertices (-2,-2), (0,3), (4,-1), and (6,4).



$$\begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix} = 2-30=-28$$

$$A = 28$$

Ex. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at (1,3,0), (-2,0,2), and (-1,3,-1).

$$\begin{bmatrix}1\\3\\0\end{bmatrix} \quad \begin{bmatrix}-2\\0\\2\end{bmatrix} \quad \begin{bmatrix}-1\\3\\-1\end{bmatrix}$$

$$\begin{vmatrix}1 & -2 & -1\\3 & 0 & 3\\0 & 2 & -1\end{vmatrix} = \begin{vmatrix}1 & -2 & -1\\0 & 6 & 6\\0 & 2 & -1\end{vmatrix} = 6\begin{vmatrix}1 & -2 & -1\\0 & 1 & 1\\0 & 2 & -1\end{vmatrix} = 6\begin{vmatrix}1 & -2 & -1\\0 & 1 & 1\\0 & 0 & -3\end{vmatrix}$$

$$= 6(1)(1)(-3) = -18$$

$$V = 18$$

Vector Spaces

We are going to start working with some abstract sets called <u>vector spaces</u>.

- Although everything we discuss can apply to vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, we will also be more general
- On the next slide, **u**, **v**, and **w** are vectors in vector space *V* and *c* and *d* can be any real number.

Def. A vector space contains objects (called vectors) on which are defined two operations, addition and scalar multiplication, which are subject to 10 axioms (rules):

1) Closed under addition → $\mathbf{u} + \mathbf{v}$ is in V
2) Closed under scalar multiplication → $c\mathbf{u}$ is in V
3) Addition is commutative → $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
4) Addition is associative → $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
5) Zero vector → There is $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
6) Opposite vector → There is $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
7) Distributive → $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8) Distributive → $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9) Scalar multiplication is associative → $c(d\mathbf{u}) = (cd)\mathbf{u}$
10) Scalar multiplication by 1 → $1\mathbf{u} = \mathbf{u}$

Ex. Define $\mathbb{S}$ as the space of all doubly infinite sequences of real numbers:

$$\{y_k\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) \qquad x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$$

Show that $\mathbb{S}$ is a vector space.

1) closed under add.

$x + y = (\ldots, x_{-2}+y_{-2}, x_{-1}+y_{-1}, x_0+y_0, x_1+y_1, x_2+y_2, \ldots) \in \mathbb{S}$ ✓

2) closed under scalar mult.

$cy = (\ldots, cy_{-2}, cy_{-1}, cy_0, cy_1, cy_2, \ldots) \in \mathbb{S}$ ✓

3) zero $\quad 0 = (\ldots, 0, 0, 0, 0, 0, \ldots) \in \mathbb{S}$ ✓

Ex. Define $\mathbb{P}_n$ as the space of polynomials of degree at most n.

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots a_n t^n \qquad \vec{q} = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n$$

Show that $\mathbb{P}_n$ is a vector space.

1) closed under add.

$$\vec{p} + \vec{q} = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n \in \mathbb{P}_n \checkmark$$

2) closed under scalar mult.

$$c\vec{p} = (ca_0) + (ca_1)t + (ca_2)t^2 + \ldots + (ca_n)t^n \in \mathbb{P}_n \checkmark$$

3) zero $\vec{0} = 0 \in \mathbb{P}_n$ ✓

Ex. Define Π_n as the space of polynomials of degree n.

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots a_n t^n,\ a_n \neq 0$$

Show that Π_n is **not** a vector space.

$\vec{0} = 0 \notin \Pi_n$

<u>Ex.</u> Define $\mathcal{F}$ as the space of real valued functions. Show that $\mathcal{F}$ is a vector space.

Ex. Define $\mathbb{Z}^2$ as the space of vectors in $\mathbb{R}^2$ with integer elements.

$$\begin{bmatrix} a \\ b \end{bmatrix}, \text{ where } a \text{ and } b \text{ are integers}$$

Show that $\mathbb{Z}^2$ is **not** a vector space.

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ d+b \end{bmatrix} \in \mathbb{Z}^2$$ closed under add. ✓

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{Z}^2$$ zero vector ✓

$$c\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix} \notin \mathbb{Z}^2$$ closed under scalar mult. ✗

Def. A subspace of a vector space V is a subset H of V that satisfies 3 rules:

1) H is closed under addition → If **u** and **v** are in H, then **u** + **v** is in H.

2) H is closed under scalar multiplication → If **u** is in H, then c**u** is in H

3) Zero vector → The zero vector of V is in H.

Every subspace is a vector space in its own right. However, since it is a subset of an already-established vector space, not all axioms need to be verified.

→ $\mathbb{P}_n$ is a subspace of $\mathcal{F}$.

→ $\mathbb{Z}^2$ is not a subspace of $\mathbb{R}^2$

Ex. $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. However, consider the set that looks and acts like $\mathbb{R}^2$.

$$H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \text{ are real} \right\}$$

$\vec{u} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} p \\ q \\ 0 \end{bmatrix}$

Show that H is a subspace of $\mathbb{R}^3$.

closed under add: $\vec{u} + \vec{v} = \begin{bmatrix} a+p \\ b+q \\ 0 \end{bmatrix} \in H$ ✓

closed under scalar mult.: $c\vec{u} = \begin{bmatrix} ca \\ cb \\ 0 \end{bmatrix} \in H$ ✓

zero: $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$ ✓

Ex. Consider the zero subspace, consisting only of the zero vector of a vector space V.

$$\{\mathbf{0}\}$$

Show that this is a subspace of V.

$\vec{0} + \vec{0} = \vec{0}$ ✓

$c\vec{0} = \vec{0}$ ✓

$\vec{0}$ ✓

Ex. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in a vector space V, show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of V.

$$H = \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 \mid c_1, c_2 \in \mathbb{R} \right\}$$

$$\vec{p} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$\vec{q} = d_1\vec{v}_1 + d_2\vec{v}_2$$

addition: $(c_1\vec{v}_1 + c_2\vec{v}_2) + (d_1\vec{v}_1 + d_2\vec{v}_2) = (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 \in H$

scalar mult. $a(c_1\vec{v}_1 + c_2\vec{v}_2) = (ac_1)\vec{v}_1 + (ac_2)\vec{v}_2 \in H$

zero $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 \in H$

- If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in a vector space V, Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is called the subspace spanned by the vectors.
- Given any subspace H, a spanning set for H is a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ such that H = Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$
- Consider $\mathbb{P}_n$ as a subspace of $\mathcal{F}$. The set $\{1, t, t^2, \ldots, t^n\}$ is a spanning set for $\mathbb{P}_n$.

Ex. Consider the set of vectors

$$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \text{ are real} \right\} = \left\{ \begin{bmatrix} a \\ -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ b \\ 0 \\ b \end{bmatrix} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Show that H is a subspace of $\mathbb{R}^4$.

H is a span, so it is a subspace