Some Subspaces

$\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Ex. Solve the equation $A\mathbf{x} = \mathbf{0}$ for $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\Longrightarrow \begin{array}{l} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{array} \Longrightarrow \begin{array}{l} x_1 = 2x_2 + x_4 - 3x_5 \\ x_2 = x_2 \\ x_3 = -2x_4 + 2x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{array} \Longrightarrow \vec{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_3}$$

The set of solutions to this system form a subspace because this set is the span of the vectors.

<u>Def.</u> The <u>null space</u> of matrix A, written Nul A, is the set of solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

Note that this only works for the homogeneous equation.

→ The solution set for $A\mathbf{x} = \mathbf{b}$ doesn't include the zero vector.

→ Also, $A\mathbf{x} = \mathbf{b}$ may have no solution

Ex. For $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, determine if $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ is in the null space of A.

If $\vec{u} \in \text{Nul } A \Rightarrow A\vec{u} = \vec{0}$

$$A\vec{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \vec{u} \in \text{Nul } A$

Another description:

Consider the linear transformation $\mathbf{x} \longmapsto A\mathbf{x}$,
Nul A is the set of all vectors that are mapped to the zero vector.

There's no obvious relation between the entries of A and the vectors in Nul A (or its spanning set).

Another subspace, which has a more obvious connection, is the column space of A.

<u>Def.</u> The <u>column space</u> of A, written Col A, is the subspace that is the span of the columns of A.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \\ 9 & 6 \end{bmatrix} \qquad \text{col}A = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix} \right\}$$

If $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n]$, a vector $\mathbf{b}$ is in Col A if
$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$

$$\{\mathbf{b} : A\mathbf{x} = \mathbf{b}\}$$

For the linear transformation $\mathbf{x} \longmapsto A\mathbf{x}$, Col A is the range.

Ex. Find a matrix A such that $W = \text{Col } A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a \text{ and } b \text{ are real numbers} \right\}$$

$$= \left\{ \begin{bmatrix} 6a \\ a \\ -7a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

Ex. Consider the 3×4 matrix A.

a. Col A is a subspace of $\mathbb{R}^k$ for what value of k?

b. Nul A is a subspace of $\mathbb{R}^k$ for what value of k?

$$\text{Col } A = \{ \vec{b} : \underset{3\times 4}{A} \underset{4\times 1}{\vec{x}} = \underset{3\times 1}{\vec{b}} \} \subseteq \mathbb{R}^3$$

$$\text{Nul } A = \{ \vec{x} : \underset{3\times 4}{A} \underset{4\times 1}{\vec{x}} = \underset{3\times 1}{\vec{0}} \} \subseteq \mathbb{R}^4$$

<u>Ex.</u> For $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, determine if $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is in the column space of A.

$\vec{x} \rightarrow \vec{u}$

$A\vec{x} = \vec{u}$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 5 \\ -5 & 9 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -2 & 5 \\ 0 & -6 & -9 & 28 \end{array}\right]$$

consist.

$\therefore \vec{u} \in \text{col } A$

Ex. For $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, find a nonzero vector in Col A and a nonzero vector in Nul A.

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$A\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -5 & 9 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 3/2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5/2 & 0 \\ 0 & 1 & 3/2 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_1 + \frac{5}{2}x_3 = 0 \\ x_2 + \frac{3}{2}x_3 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -\frac{5}{2}x_3 \\ x_2 = -\frac{3}{2}x_3 \\ x_3 = x_3 \end{array} \xrightarrow{x_3 = 2} \vec{x} = \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$$

Nul A and Col A are quite different, though we will find a connection between them next class.

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

When considering more abstract vector spaces, we discuss the linear transformation rather than the matrix.

Def. A linear transformation T from a vector space V to a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$

The <u>kernel</u> of T is the subspace of V that is mapped to the zero vector in W.

→ If T is a matrix transformation, this is the null space.

The <u>range</u> of T is the subspace of W of all vectors of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

→ If T is a matrix transformation, this is the column space.

Ex. An example of an abstract linear transformation would be the derivative.

We can use $C[a,b]$, which is the set of all continuous functions on the interval $[a,b]$.

$$x^7 \longrightarrow 7x^6$$

Kernal T: All functions that map to 0.

Kernal = {constants}

Ex. Define the linear transformation $T\colon \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \end{bmatrix}$. Find the kernel of T.

$\mathbb{P}_2 = \{a + bt + ct^2\}$

$\vec{p}(t) = a + bt + ct^2$

$\vec{p}'(t) = b + 2ct$

$T(\vec{p}) = \begin{bmatrix} a \\ b \end{bmatrix}$

Kernal is all vectors that map to $\vec{0}$

$\Longrightarrow T(\vec{p}) = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow a = 0,\ b = 0$

Kernal $= \{ct^2\} = \cancel{\text{span}}\ \{t^2\}$

Linear Independence

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is linearly dependent if there exist constants $c_1, c_2, \ldots, c_p$ (not all zero) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

→ This equation is called a linear dependence relation.

→ If the set is dependent, one of the vectors can be written as the linear combination of the others.

→ The set is linearly independent if $c_1 = c_2 = \ldots = c_p = 0$ is the only solution.

→ When we saw this before, the vectors were in $\mathbb{R}^n$ and we looked at the equation $A\mathbf{x} = \mathbf{0}$.

→ For abstract vector spaces, we can't rely on that.

Ex. In $\mathbb{P}$, determine if $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, $\mathbf{p}_3(t) = t^2$, and $\mathbf{p}_4 = \cancel{(t+3)^2}$ are linearly dependent.

t^2+6t+9

$$\underline{-9}\,1 + \underline{-6}\,t + \underline{-1}\,t^2 + \underline{\ 1\ }\,(t^2+6t+9) = 0$$

coefficents exist, so vectors are lin. dep.

Ex. In $C[0,1]$, determine if $\{\cos t, \sin t\}$ is linearly dependent.

$$\underline{0} \cos t + \underline{0} \sin t = 0$$

lin. indep.

Ex. In $C[0,1]$, show that $\{\cos t, \sin t, \sin(t+\frac{\pi}{4})\}$ is linearly dependent.

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$$\sin\left(t+\frac{\pi}{4}\right) = \sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\sin t + \frac{\sqrt{2}}{2}\cos t$$

$$\vec{v}_3 = \frac{\sqrt{2}}{2}\vec{v}_1 + \frac{\sqrt{2}}{2}\vec{v}_2$$

they are related, so vectors are lin. dep.

Def. Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a basis of H if

i. The vectors in $\mathcal{B}$ are linearly independent

ii. The vectors in $\mathcal{B}$ span H.

This could be considered the most "efficient" way to define the subspace H.

Ex. Determine if $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

i) vectors span $\mathbb{R}^3$ $\longrightarrow$ pivot in every row ✓

ii) vectors are indep. $\longrightarrow$ pivot in every column ✓

yes, they form a basis

The columns of I_n are called the <u>standard basis</u> for $\mathbb{R}^n$.

In $\mathbb{R}^3$, the standard basis vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

The set $S = \{1, t, t^2, \ldots, t^n\}$ is called the <u>standard basis</u> for $\mathbb{P}_n$.

$$3 + 7t + 4t^2 + 5t^3$$

Ex. The vectors are dependent. If H = span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, identify a basis for H.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$$

A basis is a spanning set that is as small as possible.

Ex. Let $H = \text{span}\{\underbrace{1, t, t^2}, \cancel{(t+3)^2}\}$, find a basis.

Ex. Let $H = \text{span}\{\underbrace{\cos t, \sin t}\}$, find a basis.

Ex. Let $H = \text{span}\{\underbrace{\cos t, \sin t}, \cancel{\sin(t + \frac{\pi}{4})}\}$, find a basis.

We previously found vectors that span the null space of a matrix A → this will be the basis of Nul A.

Ex. Find a basis for Nul B, where

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$B\vec{x} = \vec{0}$

$$\left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0 \end{array}$$

$$\Rightarrow \begin{array}{l} x_1 = -4x_2 - 2x_4 \\ x_2 = x_2 \\ x_3 = x_4 \\ x_4 = x_4 \\ x_5 = 0 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

basis for nul $A = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

It turns out that the pivot columns of a matrix form a basis for the column space of the matrix.

Ex. Find a basis for Col A, where

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

[This is row equiv. to B.]

basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$

Be careful to use the columns of A, not the reduced form