Change of Basis

Some problems are easier to work in a different basis.

→ We need to talk about how to change between bases.

Ex. Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ such that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$. Suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{C}}$.

$$[\vec{b}_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad [\vec{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$$\vec{x} = 3\vec{b}_1 + 1\vec{b}_2$$

$$= 3(4\vec{c}_1 + \vec{c}_2) + 1(-6\vec{c}_1 + \vec{c}_2)$$

$$= (3\cdot 4 + 1(-6))\vec{c}_1 + [(3)(1) + (1)(1)]\vec{c}_2$$

$$= 6c_1 + 4c_2 \qquad [\vec{x}]_C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\underset{C \leftarrow B}{P} \cdot [\vec{x}]_B = [\vec{x}]_C$$

$$\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

We can notice that $[\mathbf{x}]_{\mathcal{C}}$ was equal to the product of a matrix and $[\mathbf{x}]_{\mathcal{B}}$.

Further, we notice that the columns of the matrix were the $\mathcal{C}$-coordinates of the vectors in $\mathcal{B}$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathrm{P}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

This is called the <u>change-of-coordinates</u> matrix from $\mathcal{B}$ to $\mathcal{C}$

→ To change from $\mathcal{C}$ to $\mathcal{B}$, we can use the inverse

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathrm{P}} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathrm{P}}\right)^{-1}$$

In general, we can find the change-of-coordinates matrix by using

$$[P_{\mathcal{B}} \mid P_{\mathcal{C}}] \sim [I \mid \underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathrm{P}}]$$

Ex. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ be the vectors of bases $\mathcal{B}$ and $\mathcal{C}$. Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

$$\underset{B\leftarrow C}{P}: \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right] \qquad \underset{B\leftarrow C}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

$$\underset{C\leftarrow B}{P} = \frac{1}{20-18}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$$

If $[\vec{x}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, find $[\vec{x}]_C$.

$$[\vec{x}]_C = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \text{—}$$

Eigenvalues and Eigenvectors

Def. Let A be a square matrix. A number λ is an eigenvalue of A if there is a nonzero vector $\mathbf{x}$ such that

λ

$$A\mathbf{x} = \lambda\mathbf{x}$$

$\mathbf{x}$ is called an eigenvector corresponding to λ.

Ex. Show that $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -2\vec{x}$$

$$\lambda = -2$$

Ex. Show that $\lambda = 7$ is an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$A\vec{x} = \lambda\vec{x}$

$A\vec{x} = 7\vec{x}$

$A\vec{x} - 7\vec{x} = \vec{0}$

$A\vec{x} - 7I\vec{x} = \vec{0}$

$(A - 7I)\vec{x} = \vec{0}$

$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\vec{x} = \vec{0}$

$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$

$\left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow \begin{array}{l} x_1 = x_2 \\ x_2 = x_2 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\text{eigenvector} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

In the last example, we solved the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$

→ This is the null space of the matrix $A - \lambda I$

→ This is a subspace which we call the eigenspace of A corresponding to λ.

Ex. An eigenvalue of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ is λ = 2. Find a basis for the corresponding eigenspace.

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$(A - 2I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow 2x_1 - x_2 + 6x_3 = 0 \Rightarrow \begin{aligned} x_1 &= \tfrac{1}{2}x_2 - 3x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} \tfrac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thm. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent.

→ This fact will be useful later.

Ex. Show that $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

$(A - \lambda I)\vec{x} = \vec{0}$

$(A - 3I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 0 & 6 & -8 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$ ✓ $\lambda = 3$

$(A - 2I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & 6 & -8 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ✓ $\lambda = 2$

$(A - 0I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 3 & 6 & -8 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$ ✓ $\lambda = 0$

The eigenvalues of a triangular matrix are the entries on the main diagonal.

→ Note that $\lambda = 0$ is also an eigenvalue in the previous example

→ When 0 is an eigenvalue, this means that $A\mathbf{x} = 0\mathbf{x}$, or $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

→ This means that A is not invertible

Thm. Invertible Matrix Theorem

Let A be $n \times n$. The following are equivalent:

i. A is invertible
ii. A is row equivalent to I.
iii. A has n pivot positions (one in each row and column).
iv. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
v. The columns of A are linearly independent.
vi. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
vii. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.
viii. The columns of A span $\mathbb{R}^n$.
ix. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
x. The determinant of A is not zero
xi. Col $A = \mathbb{R}^n$
xii. Row $A = \mathbb{R}^n$
xiii. Nul A is the zero subspace
xiv. The number 0 is not an eigenvalue of A

For more complicated matrices A, we need a process to find eigenvalues and eigenvectors:

→ $(A - \lambda I)\mathbf{x} = 0$

→ By the Invertible Matrix Theorem, this has a nontrivial solution only when $(A - \lambda I)$ is not invertible

→ This means that $\det(A - \lambda I) = 0$

→ The equation $\det(A - \lambda I) = 0$ is called the <u>characteristic equation</u> of A.

Ex. Find the eigenvalues of $A=\begin{bmatrix}2 & 3\\ 3 & -6\end{bmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 3\\ 3 & -6-\lambda\end{vmatrix}=(2-\lambda)(-6-\lambda)-9=\lambda^2+4\lambda-21$$

$$=(\lambda+7)(\lambda-3)=0$$

$$\lambda=-7,\ 3$$

Ex. Find the eigenvalues and bases for the corresponding eigenspaces of $A=\begin{bmatrix}3&0&0\\0&2&2\\0&1&1\end{bmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix}3-\lambda&0&0\\0&2-\lambda&2\\0&1&1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}2-\lambda&2\\1&1-\lambda\end{vmatrix}=(3-\lambda)\left[(2-\lambda)(1-\lambda)-2\right]$$

$$=(3-\lambda)(\lambda^2-3\lambda)=(3-\lambda)\lambda(\lambda-3)=0 \qquad \lambda=0,3$$

$\lambda=3$: $(A-3I)\vec{x}=\vec{0}$ $\Rightarrow \left[\begin{array}{ccc|c}0&0&0&0\\0&-1&2&0\\0&1&-2&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}0&1&-2&0\\0&0&0&0\\0&0&0&0\end{array}\right] \Rightarrow x_2-2x_3=0 \Rightarrow$ $x_1=x_1$, $x_2=2x_3$, $x_3=x_3$

$$\vec{x}=\begin{bmatrix}x_1\\2x_3\\x_3\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\2\\1\end{bmatrix} \qquad \boxed{\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix}\right\}}$$

$\underline{\lambda = 0:}$ $(A - 0I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow \begin{array}{l} x_1 = 0 \\ 2x_2 + x_3 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 = x_3 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\boxed{\left\{ \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} \right\}}$$

When A is $n \times n$, the characteristic equation is degree n and is much harder to solve. To make life easier, let's review some properties of determinants:

- Let U be an echelon form of A obtained by replacements and interchanges (no scaling), and let r be the number of interchanges, then

$$\det A = \begin{cases} (-1)^r \det U & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

When A is $n \times n$, the characteristic equation is degree n and is much harder to solve. To make life easier, let's review some properties of determinants:

- $\det AB = (\det A)(\det B)$
- $\det (A^T) = \det A$
- If A is triangular, $\det A$ is the product of the entries on the main diagonal
- Row replacement doesn't change the determinant
- Row interchange changes the sign of the determinant
- Row scaling also scales the determinant by the same factor

Ex. Find the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \overset{R_2 \to -2R_1 + R_2}{=\!=\!=} \begin{vmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{vmatrix} \underset{R_3 \to \frac{-1}{3}R_2 + R_3}{=\!=\!=} \begin{vmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = (1)(-6)\left(\frac{1}{3}\right) = -2$$

Ex. Find the characteristic equation of $A = \begin{bmatrix} 3 & 6 & -8 & 4 \\ 0 & 1 & 6 & -6 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$$(3-\lambda)(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

We say that $\lambda = 3$ is an eigenvalue of multiplicity 2.

For this class, we will only focus on real valued eigenvalues.

- It should be noted, though, that eigenvalues are solutions to n^{th} degree polynomial equations, and so they can be complex

Also, in this class we will only be solving characteristic equations that are quadratic.

Def. Two matrices A and B are similar if there is some matrix P such that $A = PBP^{-1}$

We say that the transformation $A \mapsto PAP^{-1}$ is called a similarity transformation.

Thm. If two matrices are similar, then they have the same characteristic equation and, therefore, the same eigenvalues with the same multiplicities.

→ Let's prove it.

Assume $A = PBP^{-1}$

$$\det(A-\lambda I) = \det(PBP^{-1}-\lambda I) = \det(PBP^{-1}-\lambda PP^{-1})$$
$$= \det(PBP^{-1} - P\lambda I P^{-1}) = \det[P(B-\lambda I)P^{-1}]$$
$$= (\det P)[\det(B-\lambda I)](\det P^{-1}) = [\det(B-\lambda I)](\det P)(\det P^{-1})$$
$$= \det(B-\lambda I)\det(P\cdot P^{-1}) = \det(B-\lambda I)\det I = \det(B-\lambda I)$$