Inner Product

The inner product of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

is defined as

$$\vec{u} \cdot \vec{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is the dot product between **u** and **v**.

Ex. Compute $\mathbf{u} \cdot \mathbf{v}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = 6 + (-10) + 3 = -1$$

Properties of the Inner Product

1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

2) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

3) $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

4) $\mathbf{a} \cdot \mathbf{a} \geq 0$ and $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$

Def. The length (or norm) of a vector is denoted $||\mathbf{a}||$

Thm. The length of vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$

$|c| = \sqrt{c^2}$

The length of vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Note 1: $||\mathbf{a}||^2 = \mathbf{a} \cdot \mathbf{a}$

$||\vec{a}||^2 = a_1^2 + a_2^2 + a_3^2$

Note 2: $||c\mathbf{a}|| = |c|\ ||\mathbf{a}||$ ← Prove it.

$c\vec{a} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$

$||c\vec{a}|| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = \sqrt{c^2(a_1^2 + a_2^2 + a_3^2)} = |c|\ ||\vec{a}||$

Def. A unit vector is a vector whose length is 1.

→ This process is sometimes called normalization.

The unit vector in the direction of **a** is

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

Ex. Find the unit vector in the direction of $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \end{bmatrix}$

$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + (-2)^2 + 0^2} = \sqrt{4+1+4} = 3$$

$$\vec{u} = \frac{\vec{v}}{3} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \\ 0 \end{bmatrix}$$

Ex. Let W be a subspace of $\mathbb{R}^2$ spanned by $\mathbf{v} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$. Find a unit vector that is a basis for W.

$\vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

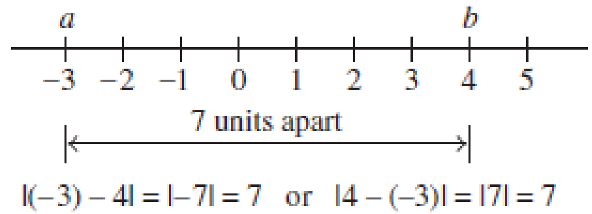
$\|\vec{w}\| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

$\vec{u} = \dfrac{\vec{w}}{\sqrt{13}} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$

$\dfrac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Distance:

In 1-D, the distance between points a and b is $|b - a|$



In higher dimensions, this can be extended to be

$$\text{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Ex. Find $\text{dist}\left(\begin{bmatrix}7\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}\right)$.

$\vec{u}$ $\vec{v}$

$$\vec{u} - \vec{v} = \begin{bmatrix}4\\-1\end{bmatrix}$$

$$\text{dist} = \|\vec{u} - \vec{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

Note this is same as using the distance formula on the points (7,1) and (3,2).

The inner product is used to find the angle between two vectors:

Thm. If θ is the angle between **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

Ex. Find the angle between $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

$$\vec{a} \cdot \vec{b} = 10 + (-6) + (-2) = 2$$

$$\|\vec{a}\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4+4+1} = 3$$

$$\|\vec{b}\| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{25+9+4} = \sqrt{38}$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$2 = 3\sqrt{38}\cos\theta$$

$$\cos\theta = \frac{2}{3\sqrt{38}}$$

Thm. Two vectors **a** and **b** are orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$.

Orthogonal = Perpendicular = Normal

Ex. Show that $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$ are orthogonal

$\vec{a}$ $\vec{b}$

$\vec{a} \cdot \vec{b} = 10 - 8 - 2 = 0$ ✓

Thm. Pythagorean Theorem

Two vectors **u** and **v** are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Ex. Verify this for $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$

$\vec{u}$ $\vec{v}$

$\begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}$ $\vec{u}+\vec{v}$

$\|\vec{u}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$\|\vec{v}\| = \sqrt{5^2 + (-4)^2 + 2^2} = \sqrt{45}$

$\|\vec{u}+\vec{v}\| = \sqrt{7^2 + 2^2 + 1^2} = \sqrt{54}$

$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$

$3^2 + \sqrt{45}^2 = \sqrt{54}^2$

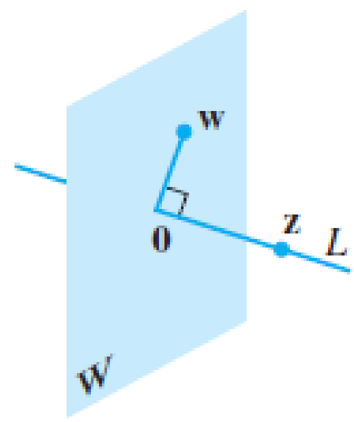
$9 + 45 = 54$ ✓

<u>Def.</u> If $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, we say $\mathbf{z}$ is <u>orthogonal to W</u>.

The set of all vectors orthogonal to W is called the <u>orthogonal complement</u> of W, and is denoted $W^{\perp}$ (called "W perpendicular" or "W perp").

- $W^{\perp}$ is itself a subspace of $\mathbb{R}^n$.

If W is a plane (through the origin) in $\mathbb{R}^3$, then $W^{\perp}$ is the set of all vectors orthogonal to the plane.



- A vector **x** is in $W^\perp$ if and only if **x** is orthogonal to every vector in a set that spans W.

→ Prove it.

Assume $W = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$= \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3\}$

Assume $\vec{x}$ is orthog. to $\vec{v}_1, \vec{v}_2, \vec{v}_3$,

$\vec{x}\cdot\vec{v}_1 = 0 \quad \vec{x}\cdot\vec{v}_2 = 0 \quad \vec{x}\cdot\vec{v}_3 = 0$

Show $\vec{x}$ is orthog. to $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$$\vec{x}\cdot\vec{w} = \vec{x}\cdot(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1(\vec{x}\cdot\vec{v}_1) + c_2(\vec{x}\cdot\vec{v}_2) + c_3(\vec{x}\cdot\vec{v}_3) = 0$$

(each $\vec{x}\cdot\vec{v}_i = 0$)

$\therefore$ $\vec{x}$ is orthog. to $\vec{w}$

$\therefore$ $\vec{x} \in W^\perp$

Ex. Is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $W^{\perp}$ where $W = \text{span}\left\{ \underset{\vec{w}_1}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \underset{\vec{w}_2}{\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}} \right\}$?

$\vec{x} \cdot \vec{w}_1 = 1 - 1 + 0 = 0$ ✓

$\vec{x} \cdot \vec{w}_2 = 2 + 0 + {}^{-}2 = 0$ ✓

$\therefore \vec{x} \in W^{\perp}$

Ex. Evaluate $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$

<u>Note:</u> $$A\mathbf{x} = \begin{bmatrix} (Row_1 A)\cdot\mathbf{x} \\ (Row_2 A)\cdot\mathbf{x} \\ \vdots \\ (Row_n A)\cdot\mathbf{x} \end{bmatrix}$$

→ If $A\mathbf{x} = \mathbf{0}$ (meaning $\mathbf{x}$ is in Nul A) then $(\text{Row}_k A) \cdot \mathbf{x} = 0$

→ $\mathbf{x}$ is orthogonal to every row of A

→ $\mathbf{x}$ is in $(\text{Row } A)^{\perp}$

$$(\text{Row } A)^{\perp} = \text{Nul } A$$

Also, Col A = Row (A^{T})

→ $(\text{Col } A)^{\perp} = (\text{Row } A^{T})^{\perp} = \text{Nul } A^{T}$

Def. A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ of vectors in $\mathbb{R}^n$ is called an orthogonal set if each pair of distinct vectors in the set is orthogonal.

→ $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ as long as $i \neq j$

Ex. Show that $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$ is an orthogonal set.

$\vec{u} \quad \vec{v} \quad \vec{w}$

$\vec{u} \cdot \vec{v} = -3 + 2 + 1 = 0$

$\vec{u} \cdot \vec{w} = -3 - 4 + 7 = 0$ ✓

$\vec{v} \cdot \vec{w} = 1 - 8 + 7 = 0$

An orthogonal set is linearly independent.

→ Let's prove it.

Assume $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthog. set

If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, show $c_1, c_2, c_3 = 0$.

$\vec{v}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = \vec{v}_1 \cdot \vec{0}$

$c_1(\vec{v}_1 \cdot \vec{v}_1) + c_2(\vec{v}_1 \cdot \vec{v}_2) + c_3(\vec{v}_1 \cdot \vec{v}_3) = 0$

($\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$)

$c_1(\vec{v}_1 \cdot \vec{v}_1) = 0$

($\vec{v}_1 \cdot \vec{v}_1 \neq 0$)

$c_1 = 0$ ✓

Def. An orthogonal basis is a basis for a subspace that is also an orthogonal set.

→ The standard basis of $\mathbb{R}^n$, $\left\{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right\}$ is an orthogonal basis.

→ An orthogonal basis is useful because it's easy to compute the weights.

Thm. If $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_p}\}$ is an orthogonal basis of W, and if $\mathbf{y}$ is in W ($\mathbf{y} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_p\mathbf{b}_p$), then the weights are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{b}_j}{\mathbf{b}_j \cdot \mathbf{b}_j}$$

→ This makes it easy to find the coordinate vector

$$[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

→ Let's prove it.

$$\vec{y} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3$$

$$\vec{b}_1 \cdot \vec{y} = \vec{b}_1 \cdot (c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3)$$

$$\vec{b}_1 \cdot \vec{y} = c_1 (\vec{b}_1 \cdot \vec{b}_1) + c_2 \underbrace{(\vec{b}_1 \cdot \vec{b}_2)}_{=0} + c_3 \underbrace{(\vec{b}_1 \cdot \vec{b}_3)}_{=0}$$

$$\frac{\vec{b}_1 \cdot \vec{y}}{\vec{b}_1 \cdot \vec{b}_1} = c_1$$

Ex. Express $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors

in the orthogonal basis $S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$.

$\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$

$$c_1 = \frac{\vec{y}\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1} = \frac{18+1-8}{9+1+1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\vec{y}\cdot\vec{b}_2}{\vec{b}_2\cdot\vec{b}_2} = \frac{-6+2-8}{1+4+1} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\vec{y}\cdot\vec{b}_3}{\vec{b}_3\cdot\vec{b}_3} = \frac{-6-4-56}{1+16+49} = \frac{-66}{66} = -1$$

$$\vec{y} = 1\vec{b}_1 - 2\vec{b}_2 - 1\vec{b}_3$$

Without this trick, we have to solve the system of equations.

Given a vector $\mathbf{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\mathbf{y}$ into the sum of two vectors: one that is in the direction of $\mathbf{u}$ and the other that is orthogonal to $\mathbf{u}$:

$$\vec{y} \cdot \vec{u} = \|\vec{y}\|\|\vec{u}\| \cos\theta$$

$$\cos\theta = \frac{\vec{y} \cdot \vec{u}}{\|\vec{y}\|\,\|\vec{u}\|}$$

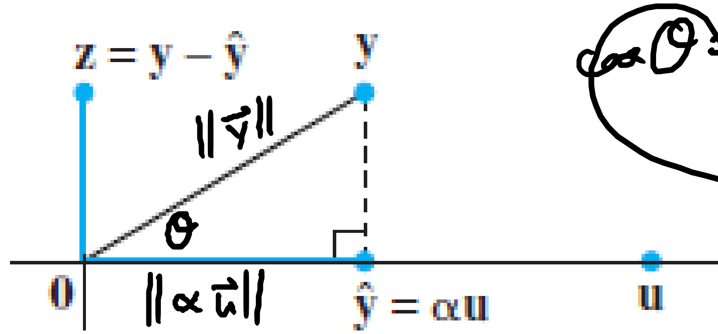
$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

$\hat{\mathbf{y}}$ is in the direction of $\mathbf{u}$, so $\hat{\mathbf{y}} = \alpha\mathbf{u}$.

→ This is called the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$

$\mathbf{z}$ is called the component of $\mathbf{y}$ orthogonal to $\mathbf{u}$

→ Let's find a formula for α



$$\cos\theta = \frac{\|\alpha\vec{u}\|}{\|\vec{y}\|} = \frac{\alpha\|\vec{u}\|}{\|\vec{y}\|}$$

$$\frac{\vec{y} \cdot \vec{u}}{\|\vec{y}\|\,\|\vec{u}\|} = \frac{\alpha\,\|\vec{u}\|}{\|\vec{y}\|}$$

$$\alpha = \frac{\vec{y} \cdot \vec{u}}{\|\vec{u}\|^2} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

If the L is the line containing $\mathbf{u}$ (all multiples of $\mathbf{u}$), we can write

$$\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

$$\vec{z} = \vec{y} - \hat{y}$$

Ex. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span{$\mathbf{u}$} and a vector orthogonal to $\mathbf{u}$.

$\hat{y}$ $\qquad\qquad\qquad\qquad$ $\vec{z}$

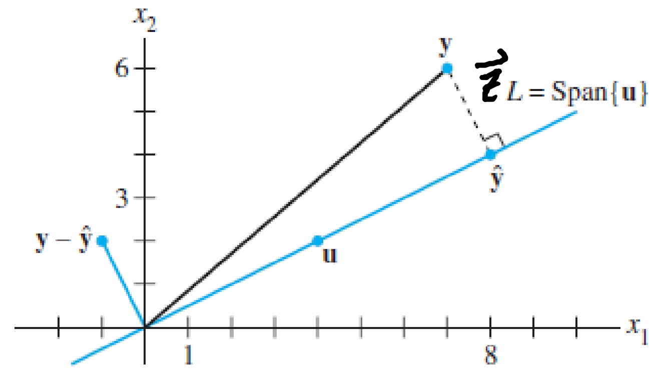
$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{28+12}{16+4}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Ex. In the previous example, find the distance from **y** to the line containing **u**.

$$\|\vec{z}\| = \|\vec{y} - \hat{y}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$



Def. An orthonormal set is an orthogonal set of unit vectors.

If a subspace is spanned by an orthonormal set, then the set is called an orthonormal basis for the subspace.

Ex. Show that $\left\{ \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix} \right\}$ is an orthonormal set.

$\vec{u}$, $\vec{v}$, $\vec{w}$

$$\vec{u} \cdot \vec{v} = \frac{-3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$$

$$\vec{u} \cdot \vec{w} = \frac{-3}{\sqrt{726}} + \frac{-4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0$$

$$\vec{v} \cdot \vec{w} = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$$

$$\|\vec{u}\| = \sqrt{\frac{9}{11} + \frac{1}{11} + \frac{1}{11}} = \sqrt{\frac{11}{11}} = 1$$

$$\|\vec{v}\| = \sqrt{\frac{1}{6} + \frac{4}{6} + \frac{1}{6}} = \sqrt{\frac{6}{6}} = 1$$

$$\|\vec{w}\| = \sqrt{\frac{1}{66} + \frac{16}{66} + \frac{49}{66}} = \sqrt{\frac{66}{66}} = 1$$

This is the earlier orthogonal set where each vector is made into a unit vector.

Thm. A matrix U (not necessarily square) has orthonormal columns if and only if $U^TU = I$.

$\vec{u}_1, \vec{u}_2, \vec{u}_3$ are orthonormal

→ Prove it

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$$

$$U^T = \begin{bmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \\ \vec{u}_3^{\,T} \end{bmatrix}$$

$$U^TU = \begin{bmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \\ \vec{u}_3^{\,T} \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \vec{u}_1 \cdot \vec{u}_3 \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_2 & \vec{u}_2 \cdot \vec{u}_3 \\ \vec{u}_3 \cdot \vec{u}_1 & \vec{u}_3 \cdot \vec{u}_2 & \vec{u}_3 \cdot \vec{u}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thm. Let U be a matrix with orthonormal columns.

i. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

ii. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

iii. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

→ This means that the transformation $\mathbf{x} \mapsto U\mathbf{x}$ preserves length and orthogonality

If U is a square matrix with orthonormal columns, then

$$U^T U = I$$

→ $U^T = U^{-1}$

→ We call this an orthogonal matrix → columns are orthonormal

→ In this case, the rows of U are orthonormal as well

$$\begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$$