Orthogonal Projections

Last class, we projected a vector **y** onto a line that was the span of a vector **u** → a subspace with dimension 1

Today, we will discuss projecting a vector **y** onto a subspace that has a dimension greater than 1

Consider $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4,\mathbf{u}_5\}$, an orthogonal basis of $\mathbb{R}^5$ and the vector $\mathbf{y}$ in $\mathbb{R}^5$.

Consider the subspace $W = \text{span}\{\mathbf{u}_1,\mathbf{u}_2\}$.

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5$$

$$\mathbf{y} = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2) + (c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5) = \vec{z}_2$$

$$\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$$

→ $\mathbf{z}_1$ is in W, let's show $\mathbf{z}_2$ is in $W^\perp$

Show $\vec{z}_2$ is orthog. to $\vec{u}_1$ and $\vec{u}_2$

$$\vec{z}_2 \cdot \vec{u}_1 = (c_3\vec{u}_3 + c_4\vec{u}_4 + c_5\vec{u}_5)\cdot\vec{u}_1 = c_3(\vec{u}_3\cdot\vec{u}_1) + c_4(\vec{u}_4\cdot\vec{u}_1) + c_5(\vec{u}_5\cdot\vec{u}_1) = 0$$

$$\vec{z}_2 \cdot \vec{u}_2 = \quad \cdot \quad \cdot \quad \cdot \quad - \quad - \quad \cdot \quad - \quad \cdot \quad \cdot = 0$$

This means that $W^\perp = \text{span}\{\mathbf{u}_3,\mathbf{u}_4, \mathbf{u}_5\}$

Thm. Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Every vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

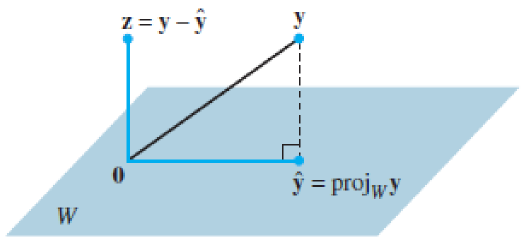
where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

$\hat{\mathbf{y}}$ is called the orthogonal projection of $\mathbf{y}$ onto W and is sometimes written $\text{proj}_W \mathbf{y}$.



Ex. Let $\left\{ \underbrace{\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\vec{u}_2} \right\}$ be an orthogonal basis for W. Write $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as the sum of a vector in W and a vector in $W^\perp$.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \frac{2+10-3}{4+25+1}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{-2+2+3}{4+1+1}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .6 \\ 1.5 \\ -.3 \end{bmatrix} + \begin{bmatrix} -1 \\ .5 \\ .5 \end{bmatrix}$$

$$= \begin{bmatrix} -.4 \\ 2 \\ .2 \end{bmatrix} \in W$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -.4 \\ 2 \\ .2 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 0 \\ 2.8 \end{bmatrix} \in W^\perp$$

$$\vec{y} = \hat{y} + \vec{z} = \begin{bmatrix} -.4 \\ 2 \\ .2 \end{bmatrix} + \begin{bmatrix} 1.4 \\ 0 \\ 2.8 \end{bmatrix}$$

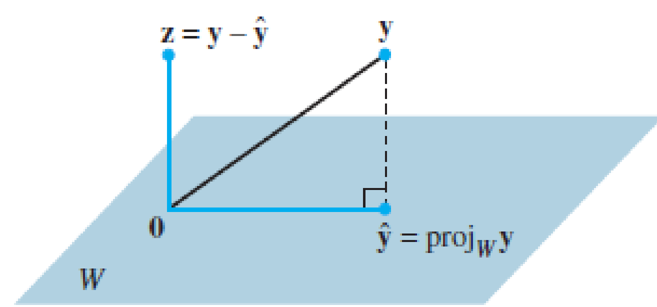
Thm. Best Approximation Theorem

$\hat{\mathbf{y}}$ is the vector in W that is closest to $\mathbf{y}$, in the sense that, for any vector $\mathbf{v}$ in W,

$$\|\mathbf{y} - \hat{\mathbf{y}}\| \leq \|\mathbf{y} - \mathbf{v}\|$$

$\hat{\mathbf{y}}$ is called the best approximation of $\mathbf{y}$ by elements of W.

Because we haven't discussed the basis of W, this means that $\hat{\mathbf{y}}$ is the same no matter what basis is used for W.



Ex. Find the distance from $\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$ to $W = \text{span}\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$. orthog.

$\|\vec{z}\|$

$\vec{u}_1 \quad \vec{u}_2$

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \frac{-5+10+10}{25+4+1}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{-1-10-10}{1+4+1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1 \\ .5 \end{bmatrix} + \begin{bmatrix} -3.5 \\ -7 \\ 3.5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$$\text{dist.} = \|\vec{z}\| = \sqrt{0+3^2+6^2} = \boxed{\sqrt{45}}$$

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\cancel{\mathbf{u}_1\cdot\mathbf{u}_1}}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\cancel{\mathbf{u}_2\cdot\mathbf{u}_2}}\mathbf{u}_2 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\cancel{\mathbf{u}_p\cdot\mathbf{u}_p}}\mathbf{u}_p$$

If we define $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_p]$, then $\hat{\mathbf{y}} = UU^T\mathbf{y}$

→ Prove it

$$U^T = \begin{bmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \\ \vdots \\ \vec{u}_p^{\,T} \end{bmatrix}$$

$$\hat{y} = U\cdot U^T\vec{y} = U\begin{bmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \\ \vdots \\ \vec{u}_p^{\,T} \end{bmatrix}\vec{y} = U\begin{bmatrix} \vec{u}_1\cdot\vec{y} \\ \vec{u}_2\cdot\vec{y} \\ \vdots \\ \vec{u}_p\cdot\vec{y} \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p \end{bmatrix}\begin{bmatrix} \vec{u}_1\cdot\vec{y} \\ \vec{u}_2\cdot\vec{y} \\ \vdots \\ \vec{u}_p\cdot\vec{y} \end{bmatrix}$$

$$= (\vec{u}_1\cdot\vec{y})\vec{u}_1 + (\vec{u}_2\cdot\vec{y})\vec{u}_2 + \cdots + (\vec{u}_p\cdot\vec{y})\vec{u}_p$$

Ex. Let $W = \text{span}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ (note its orthogonal).

Find a matrix A such that $\underbrace{\text{proj}_W \mathbf{y}}_{\hat{y}} = \underbrace{A}_{UU^T}\mathbf{y}$ for any vector $\mathbf{y}$.

$$\vec{u}_1 = \frac{1}{\sqrt{1+0+1}}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

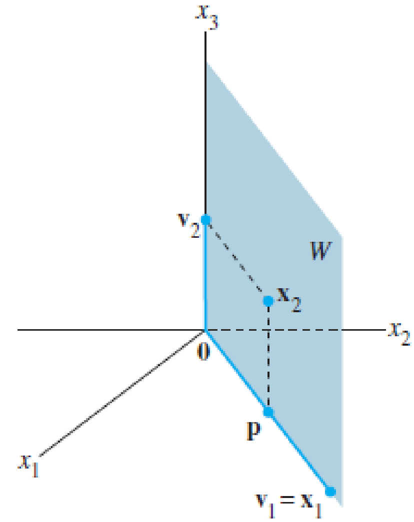
$$A = UU^T = \begin{bmatrix}1/\sqrt{2} & 0\\0 & 1\\-1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 0 & -1/\sqrt{2}\\0 & 1 & 0\end{bmatrix} = \begin{bmatrix}1/2 & 0 & -1/2\\0 & 1 & 0\\-1/2 & 0 & 1/2\end{bmatrix}$$

We've seen that it's useful to have an orthogonal basis

→ If we are given some other basis, we can find an orthogonal basis using the Gram-Schmidt Process

$$\boxed{\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \text{ a basis of } W} \longrightarrow \boxed{\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \text{ an orthogonal basis of } W}$$

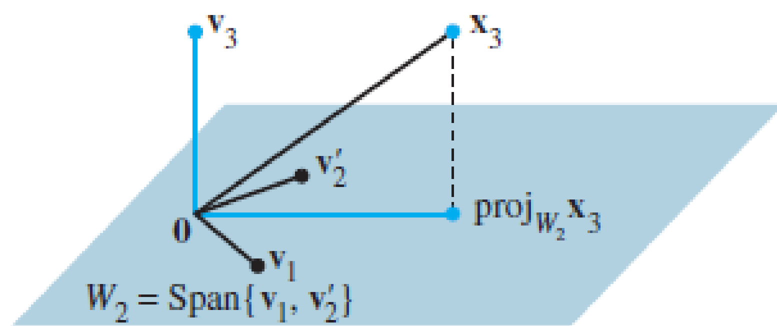
Gram-Schmidt Process



$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \overbrace{\text{proj}_{\text{span}\{\mathbf{v}_1\}}\mathbf{x}_2}^{\hat{x}_2} = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \overbrace{\text{proj}_{\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{x}_3}^{\hat{x}_3} = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$



Ex. Let $W = \text{span}\left\{\begin{bmatrix}3\\6\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$, find an orthogonal basis.

$$\underset{\vec{x}_1}{\begin{bmatrix}3\\6\\0\end{bmatrix}} \qquad \underset{\vec{x}_2}{\begin{bmatrix}1\\2\\2\end{bmatrix}}$$

$$\vec{v}_1 = \begin{bmatrix}3\\6\\0\end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{v}_1 \cdot x_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix} - \frac{3+12+0}{9+36+0}\begin{bmatrix}3\\6\\0\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

$$\left\{\begin{bmatrix}3\\6\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix}\right\}$$

<u>Ex.</u> Let $W = \text{span}\left\{ \underset{\vec{x}_1}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}, \underset{\vec{x}_2}{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}}, \underset{\vec{x}_3}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}} \right\}$, find an orthogonal basis.

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix} \longrightarrow \vec{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} - \begin{bmatrix}-1/2\\1/6\\1/6\\1/6\end{bmatrix} = \begin{bmatrix}0\\-2/3\\1/3\\1/3\end{bmatrix} \rightarrow \vec{v}_3 = \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}$$

$\vec{v}_1 \cdot \vec{v}_2 = 0$ ✓ $\quad \vec{v}_2 \cdot \vec{v}_3 = 0$ ✓

$\vec{v}_1 \cdot \vec{v}_3 = 0$ ✓

Ex. Let $W = \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$, find an orthonormal basis.

$\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}$

$\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$ $\vec{u}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \dfrac{1}{\sqrt{12}}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-3/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\end{bmatrix}$

$\vec{u}_3 = \dfrac{\vec{v}_3}{\|\vec{v}_3\|} = \dfrac{1}{\sqrt{6}}\begin{bmatrix}0\\-2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$

Ex. Let $W = \text{Nul}\begin{bmatrix} 1 & -1 & -2 \\ 2 & -2 & -4 \\ 4 & -4 & -8 \end{bmatrix}$, find the point in W that is closest to $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and find the distance from $\mathbf{y}$ to W.

$\hat{y}$ $\quad \|\vec{z}\|$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 2 & -2 & -4 & 0 \\ 4 & -4 & -8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 - x_2 - 2x_3 = 0 \Rightarrow \begin{array}{l} x_1 = x_2 + 2x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{n}_1} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{n}_2}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \vec{n}_2 - \frac{\vec{n}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}_2} \right\}$$

$$\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\hat{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \frac{0}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$\|\vec{z}\| = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$$