Solution Sets of Linear Systems

The linear system $A\mathbf{x} = \mathbf{0}$ is called homogeneous.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

This system always has at least 1 solution, where all the x's are 0. This is called the trivial solution.

Thm. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

→ So the homogeneous solution has either one trivial solution or infinitely many solutions.

Ex. Describe the solution set of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

1 free variable resulted in a line in $\mathbb{R}^3$.

Ex. Describe the solution set of $10x_1 - 3x_2 - 2x_3 = 0$

2 free variables resulted in a plane in $\mathbb{R}^3$.

If A has no free variables:

- Trivial solution
- The point $\mathbf{0}$ in $\mathbb{R}^3$

If A has 1 free variable:

- A line in $\mathbb{R}^3$ that passes through the origin
- Can be described parametrically by $\mathbf{x} = t\mathbf{v}_1$.

If A has 2 free variables:

- A plane in $\mathbb{R}^3$ that passes through the origin
- Can be described parametrically by $\mathbf{x} = s\mathbf{v}_1 + t\mathbf{v}_2$.

→ Note this represents Span$\{\mathbf{v}_1, \mathbf{v}_2\}$

When we write our solution sets in this form, it is called the parametric vector form.

If $\mathbf{b} \neq \mathbf{0}$, the linear system $A\mathbf{x} = \mathbf{b}$ is called non-homogeneous.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Ex. Describe the solution set of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

$A\mathbf{x} = \mathbf{b}$ has no solutions if:

- $A\mathbf{x} = \mathbf{b}$ is inconsistent

$A\mathbf{x} = \mathbf{b}$ has 1 solution if:

- The corresponding homogeneous system had only the trivial solution

$A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if:

- The corresponding homogeneous system had infinitely many solutions
- Solutions would be 1 vector plus a linear combination of vectors that satisfy the corresponding homogeneous system.
- $\mathbf{x} = \mathbf{p} + t\mathbf{v}_1$ → a line not through the origin
- $\mathbf{x} = \mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2$ → a plane not through the origin

Prove the previous result:

Linear Independence

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is linearly dependent if there exist constants $x_1, x_2, \ldots, x_p$ (not all zero) such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

→ This equation is called a linear dependence relation.

→ The set is linearly independent if $x_1 = x_2 = \ldots = x_p = 0$ is the only solution.

<u>Ex.</u> Determine if the vectors are dependent. Find a linear dependence relation.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

→ Note this is the same as our homogeneous equation $A\mathbf{x} = \mathbf{0}$, where the vectors are the columns of A.

Thm. The following are equivalent:

i. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

ii. The columns of A are linearly independent

iii. The linear system with augmented matrix $[A \quad \mathbf{0}]$ has no free variables

iv. A has a pivot in each column

Ex. Determine if the vectors are dependent.

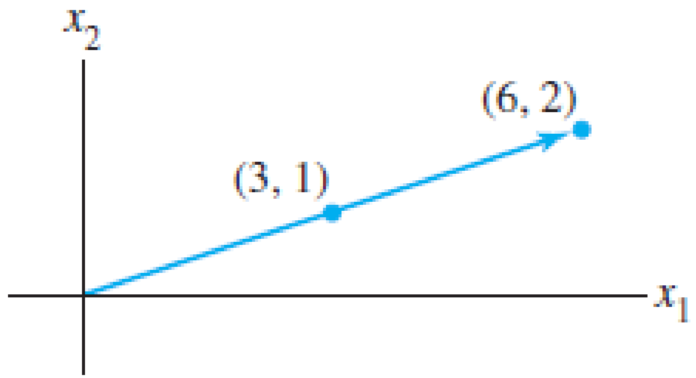
$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

Ex. Determine if the vectors are dependent.

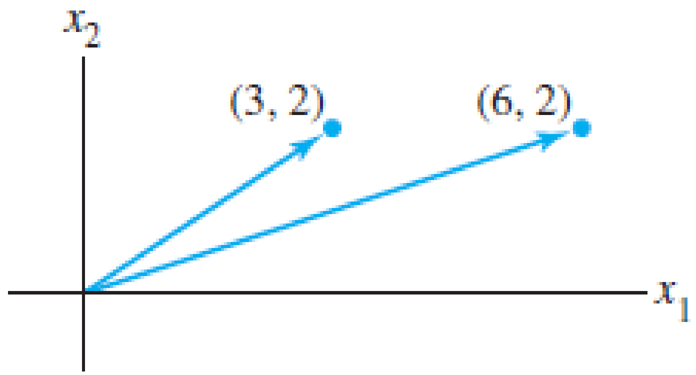
a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Two vectors are linearly dependent if one is a multiple of the other.



Linearly dependent



Linearly independent

Note: This doesn't work for more than 2 vectors!

Thm. A set of two or more vectors is linearly dependent if and only if at least one is a linear combination of the others.

Ex. Describe the set spanned by **u** and **v**.
Explain why a vector **w** is in Span{**u**,**v**} if and only if {**u**, **v**, **w**} is linearly dependent.

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$

Thm. If a set contains more vectors than there are entries in each vector, then the set is dependent.

Ex. Show that the set is dependent. $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

Thm. If a set contains the zero vector, then the set is dependent.

<u>Ex.</u> Determine if the set is dependent.

a. $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix} \right\}$