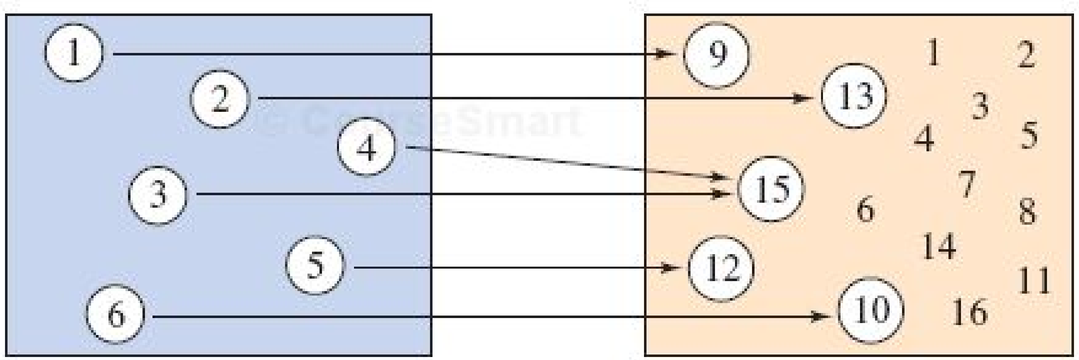
Intro to Linear Transformations

Def. A function f from set A to set B is a relation that assigns to each element x in set A exactly one element y in set B.



Set A Set B

Domain Codomain

$$\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

We can think of A as transforming $\mathbf{x}$ in $\mathbb{R}^4$ to $\mathbf{b}$ in $\mathbb{R}^2$.

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$\mathbb{R}^n$ is the domain

$\mathbb{R}^m$ is the codomain

The set of all $T(\mathbf{x})$ is called the range

→ The range is a subset of the codomain

The rest of this section will focus on mappings associated with matrix multiplication

$$\mathbf{x} \mapsto A\mathbf{x}$$

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

a. If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $T(\mathbf{u})$.

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

b. If $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, find an $\mathbf{x}$ whose image under T is $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

Was this answer unique?

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

c. If $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, find an $\mathbf{x}$ whose image under T is $\mathbf{c}$.

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

d. Find all $\mathbf{x}$ that are mapped into the zero vector.

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

Ex. Find the image of **x** under the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

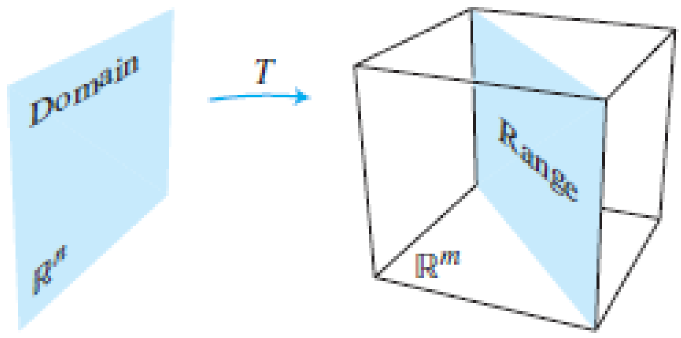
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$$

This projects the point onto the x_1x_2-plane.

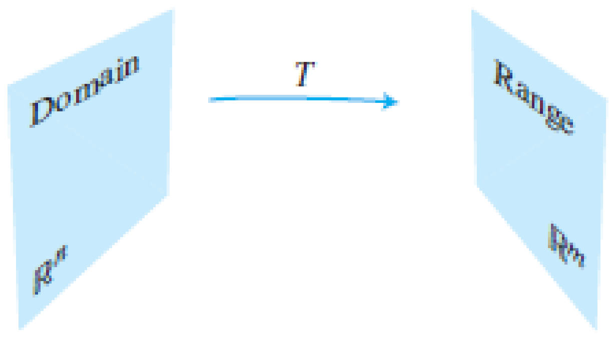
A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every **b** in $\mathbb{R}^m$ is the image of *at least* one **x** in $\mathbb{R}^n$.

→ The range makes up the entire codomain

→ Every vector in $\mathbb{R}^m$ is the output at least once



T is *not* onto $\mathbb{R}^m$



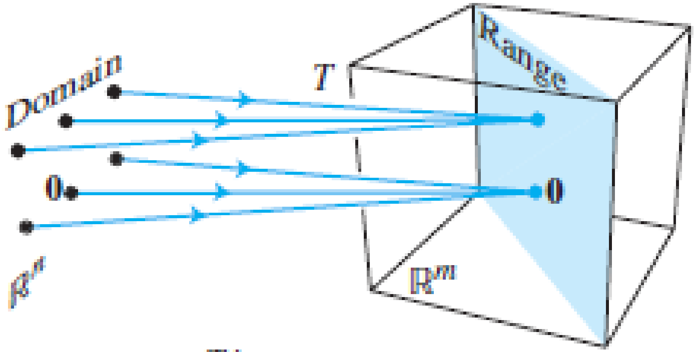
T is onto $\mathbb{R}^m$

A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is <u>one-to-one</u> if every **b** in $\mathbb{R}^m$ is the image of *at most* one **x** in $\mathbb{R}^n$.

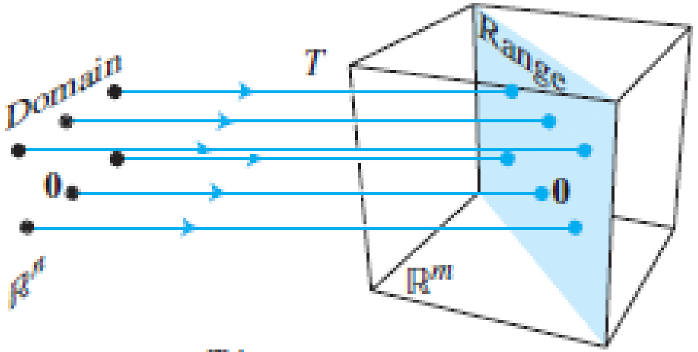
→ Every vector in the range is an output exactly once

→ Not all vectors in $\mathbb{R}^m$ are outputs

→ $T(\mathbf{x})$ has either a unique solution or no solution



T is *not* one-to-one



T is one-to-one

Ex. Define $T: \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T one-to-one?

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

We remember properties of vector/matrix/scalar addition and multiplication:

Distributive: $A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Commutative: $A(c\mathbf{u}) = cA(\mathbf{u})$

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

These lead to the properties of a <u>linear</u> transformation T.

For any linear transformation,

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

In particular, $T(\mathbf{0}) = \mathbf{0}$.

→ This can be generalized to be true for any number of vectors. This is called the superposition principle.

Ex. Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = 3\mathbf{x}$. Show that T is a linear transformation.

What does this transformation represent graphically?

<u>Ex.</u> Define T: $\mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$.

Find $T(\mathbf{u})$:

a) $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

b) $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

What does this transformation represent graphically?

Matrix of a Linear Transformation

We have been talking about different linear transformations, not just ones that are matrix multiplication.

In fact, all linear transformations can be represented by a matrix multiplication.

To find the matrix, we will be using the columns of I_n, which we will call $\mathbf{e}_1$, $\mathbf{e}_2$, etc.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These are called the standard basis vectors of $\mathbb{R}_3$.

Ex. Suppose T is a linear transformation such that $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. Describe the image of an arbitrary $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Thm. If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, there is a unique $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$.

→ The columns of A will be the transformation of the columns of I. In other words:

$$A = [T(\mathbf{e}_1) \quad \dots \quad T(\mathbf{e}_n)]$$

→ This is called the standard matrix for the linear transformation.

→ Please note mapping $\mathbb{R}^n \to \mathbb{R}^m$ requires a matrix that is $m \times n$.

Ex. Find the standard matrix for the transformation that rotates each point in $\mathbb{R}^2$ counterclockwise about the origin through an angle φ.

p. 73-75 has the standard matrices for several common geometric linear transformations.

→ Even more transformations come from the composition of transformations.

Ex. Define T: $\mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T one-to-one?

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Thm. Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. The following are equivalent:

i. T is one-to-one.

ii. A has a pivot in each column.

iii. A has no free variables.

iv. The columns of A are linearly independent.

v. The equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

→ This links us with all of the equivalent statements from last class.

Thm. Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. The following are equivalent:

i. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

ii. A has a pivot in each row.

iii. The columns of A span $\mathbb{R}^m$.

Ex. Let $T(x_1,x_2) = (3x_1 + x_2,\ 5x_1 + 7x_2,\ x_1 + 3x_2)$.
Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? Is T one-to-one?

Ex. Let T: $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation such that $T(x_1,x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (0,-1,4)$.