Matrix Operations

$$\begin{bmatrix} 3 & 5 & -2 & 3 \\ 1 & 0 & 9 & 7 \\ -4 & 8 & 6 & -3 \end{bmatrix}$$

The 5 is entry a_{12} because it is in the 1st row and 2nd column

Entries a_{11}, a_{22}, etc. are called the main diagonal

A diagonal matrix is a square ($n \times n$) matrix whose nondiagonal entries are 0.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Two matrices are equal if they have the same order and if the corresponding entries are equal

Adding and subtracting matrices means performing the operations on corresponding entries

- The matrices must have the same order, and the result will also have that order

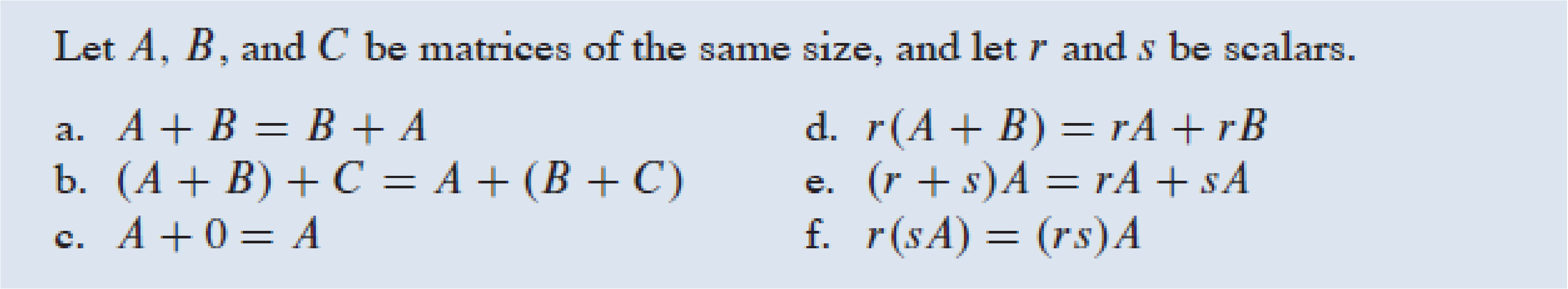
Ex.

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 1 & 4 \\ 0 & 2 & -5 \end{bmatrix}$

Scalar multiplication means multiplying a matrix by a constant

- We do this by multiplying each entry by the constant



Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Ex. Let $A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

a. $3A$

b. $3A - B$

When multiplying two matrices, we take a row from the first matrix and multiply it by a column from the second matrix

The orders have to match up:

$$\underset{4\times 3}{A} \times \underset{3\times 7}{B} = \underset{4\times 7}{AB}$$

Ex.

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix}$$

$$(1)(-2) + (0)(1) + (3)(-1) = -5$$

$$(1)(4) + (0)(0) + (3)(1) = 7$$

$$(2)(-2) + (-1)(1) + (-2)(-1) = -3$$

$$(2)(4) + (-1)(0) + (-2)(1) = 6$$

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -3 & 1 & 6 \\ 4 & 8 & 3 \end{bmatrix}$

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$
for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 4 & 8 \end{bmatrix}$

Ex. $\begin{bmatrix} -3 & 1 \\ 4 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Cautions

i. In general, $AB \neq BA$. If fact, depending on the sizes, both products may not be possible.

ii. Cancellation laws do not hold. In other words, if $AB = AC$, it may not be true that $B = C$.

iii. If $AB = 0$, it may not be true that $A = 0$ or $B = 0$.

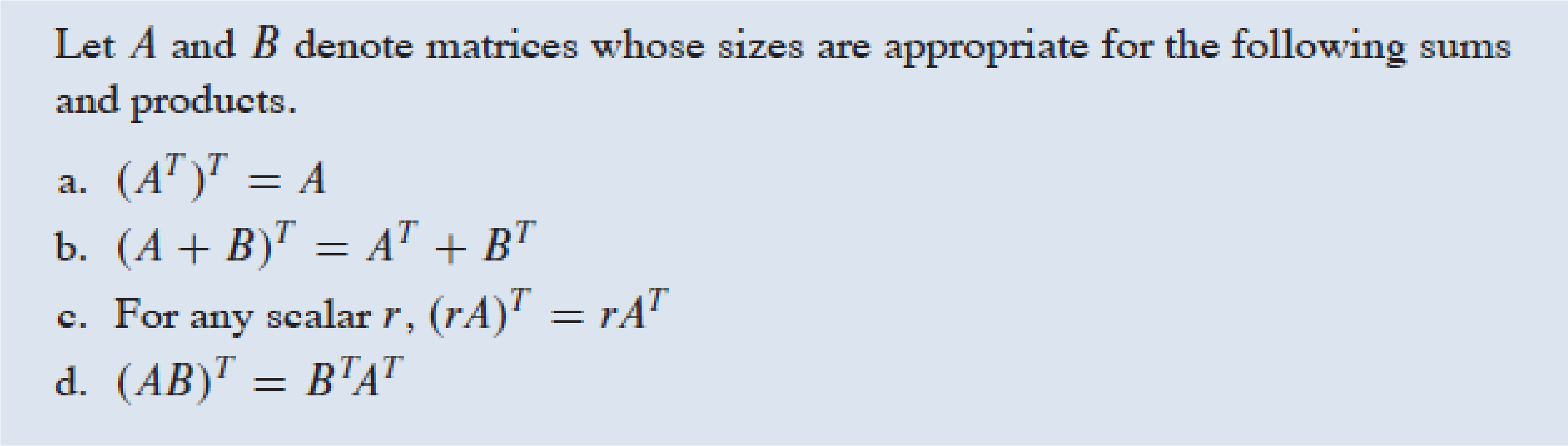
→ A^{T} (transpose) switches a_{ij} with a_{ji}

<u>Ex.</u> Find the transpose of each matrix:

a. $\mathrm{A} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix}$

b. $B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$

c. $C = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & -2 \end{bmatrix}$



Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^TA^T$

Inverse Matrices

2 and ½ are multiplicative inverses because

$$2 \times \tfrac{1}{2} = 1$$

The inverse of matrix *A* is written A^{-1}, and

$$AA^{-1} = I \text{ and } A^{-1}A = I$$

where *I* is the identity matrix

If a matrix has an inverse, we say that it is invertible

- Otherwise, we say that it is singular
- Only square matrices can be invertible

Ex. Show that $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ are inverses

For a 2 × 2 matrix, there's a quick way to find the inverse:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The quantity $ad - bc$ is called the determinant of the matrix:

$$\det A = ad - bc$$

We’ll come back to this in the future…

Ex. Find the inverse

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

b. $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

To solve the equation $ax = b$, we multiply by the multiplicative inverse $\frac{1}{a}$:

$$ax = b$$

$$\frac{1}{a}ax = \frac{1}{a}b$$

$$x = \frac{b}{a}$$

To solve a matrix equation, we do the same

$$A\mathbf{x} = B$$

$$A^{-1}A\mathbf{x} = A^{-1}B$$

$$\mathbf{x} = A^{-1}B$$

Ex. Solve the system $\begin{cases} 3x_1 + 4x_2 = -2 \\ 5x_1 + 3x_2 = 4 \end{cases}$

Thm.

i. $(A^{-1})^{-1} = A$

ii. $(AB)^{-1} = B^{-1}A^{-1}$

iii. $(A^T)^{-1} = (A^{-1})^T$

Let's prove these results.

For larger matrices, to find an inverse matrix we use row operations

- Create the matrix $[A \quad I]$
- Perform row operations to make the left side into I
- The result will be $[I \quad A^{-1}]$

Ex. Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

Thm. Invertible Matrix Theorem

Let A be $n \times n$. The following are equivalent:

i. A is invertible
ii. A is row equivalent to I.
iii. A has n pivot positions (one in each row and column).
iv. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
v. The columns of A are linearly independent.
vi. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
vii. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.
viii. The columns of A span $\mathbb{R}^n$.
ix. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

$$\begin{tikzcd}
 & A \text{ is invertible} \arrow[d, Leftrightarrow] & \\
 & A \text{ is row equiv. to } I \arrow[d, Leftrightarrow] & \\
 & A \text{ has } n \text{ pivot points} \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
\text{Pivot in every row} \arrow[d, Leftrightarrow] & & \text{Pivot in every column} \arrow[d, Leftrightarrow] \\
A\mathbf{x} = \mathbf{b} \text{ has at least one solution} \arrow[d, Leftrightarrow] & & A\mathbf{x} = \mathbf{b} \text{ has at most one solution} \arrow[d, Leftrightarrow] \\
\text{Columns span } \mathbb{R}^n \arrow[d, Leftrightarrow] & & \text{Columns are linearly indep.} \arrow[d, Leftrightarrow] \\
\text{Transformation is onto} & & A\mathbf{x} = \mathbf{0} \text{ only trivial solution} \arrow[d, Leftrightarrow] \\
 & & \text{Transformation is one-to-one}
\end{tikzcd}$$

Ex. Determine if A is invertible.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

Matrices A and B are inverses if $AB = I$ and $BA = I$.

→ Transformations T and S are inverses if

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad S(T(\mathbf{x})) = \mathbf{x}$$

In fact, if A is the standard matrix for T, then A^{-1} is the standard matrix for T^{-1}.