Some Subspaces

Ex. Solve the equation $A\mathbf{x} = \mathbf{0}$ for $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

The set of solutions to this system form a subspace because this set is the span of the vectors.

<u>Def.</u> The <u>null space</u> of matrix A, written Nul A, is the set of solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

Note that this only works for the homogeneous equation.

→ The solution set for $A\mathbf{x} = \mathbf{b}$ doesn't include the zero vector.

→ Also, $A\mathbf{x} = \mathbf{b}$ may have no solution

Ex. For $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, determine if $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ is in the null space of A.

<u>Another description:</u>

Consider the linear transformation $\mathbf{x} \longmapsto A\mathbf{x}$, Nul A is the set of all vectors that are mapped to the zero vector.

There's no obvious relation between the entries of A and the vectors in Nul A (or its spanning set).

Another subspace, which has a more obvious connection, is the column space of A.

Def. The column space of A, written Col A, is the subspace that is the span of the columns of A.

If $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n]$, a vector $\mathbf{b}$ is in Col A if $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$

$$\{\mathbf{b} : A\mathbf{x} = \mathbf{b}\}$$

For the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$, Col A is the range.

Ex. Find a matrix A such that $W = \text{Col}\, A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a \text{ and } b \text{ are real numbers} \right\}$$

Ex. Consider the 3×4 matrix A.

a. Col A is a subspace of $\mathbb{R}^k$ for what value of k?

b. Nul A is a subspace of $\mathbb{R}^k$ for what value of k?

Ex. For $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, determine if $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is in the column space of A.

Ex. For $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, find a nonzero vector in Col A and a nonzero vector in Nul A.

Nul A and Col A are quite different, though we will find a connection between them next class.

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

When considering more abstract vector spaces, we discuss the linear transformation rather than the matrix.

Def. A linear transformation T from a vector space V to a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$

The <u>kernel</u> of T is the subspace of V that is mapped to the zero vector in W.

→ If T is a matrix transformation, this is the null space.

The <u>range</u> of T is the subspace of W of all vectors of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

→ If T is a matrix transformation, this is the column space.

Ex. An example of an abstract linear transformation would be the derivative.

We can use $C[a,b]$, which is the set of all continuous functions on the interval $[a,b]$.

Ex. Define the linear transformation T: $\mathsf{P}_2 \longrightarrow \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}'(0) \end{bmatrix}$. Find the kernel of T.

Linear Independence

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is linearly dependent if there exist constants $c_1, c_2, \ldots, c_p$ (not all zero) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

→ This equation is called a linear dependence relation.

→ If the set is dependent, one of the vectors can be written as the linear combination of the others.

→ The set is linearly independent if $c_1 = c_2 = \ldots = c_p = 0$ is the only solution.

→ When we saw this before, the vectors were in $\mathbb{R}^n$ and we looked at the equation $A\mathbf{x} = \mathbf{0}$.

→ For abstract vector spaces, we can't rely on that.

Ex. In P, determine if $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, $\mathbf{p}_3(t) = t^2$, and $\mathbf{p}_4 = (t + 3)^2$ are linearly dependent.

<u>Ex.</u> In $C[0,1]$, determine if $\{\cos t, \sin t\}$ is linearly dependent.

Ex. In $C[0,1]$, show that $\{\cos t, \sin t, \sin(t + \frac{\pi}{4})\}$ is linearly dependent.

Def. Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a basis of H if

i. The vectors in $\mathcal{B}$ are linearly independent

ii. The vectors in $\mathcal{B}$ span H.

This could be considered the most "efficient" way to define the subspace H.

Ex. Determine if $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

The columns of I_n are called the standard basis for $\mathbb{R}^n$.

In $\mathbb{R}^3$, the standard basis vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The set $S = \{1, t, t^2, \ldots, t^n\}$ is called the standard basis for P_n.

<u>Ex.</u> The vectors are dependent. If $H = \text{span}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, identify a basis for H.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

A basis is a spanning set that is as small as possible.

Ex. Let $H = \text{span}\{1, t, t^2, (t+3)^2\}$, find a basis.

Ex. Let $H = \text{span}\{\cos t, \sin t\}$, find a basis.

Ex. Let $H = \text{span}\{\cos t, \sin t, \sin(t + \frac{\pi}{4})\}$, find a basis.

We previously found vectors that span the null space of a vector A → this will be the basis of Nul A.

Ex. Find a basis for Nul B, where

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It turns out that the pivot columns of a matrix form a basis for the column space of the matrix.

Ex. Find a basis for Col A, where

[This is row equiv. to B.]

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

Be careful to use the columns of A, not the reduced form