Coordinate Systems

Consider the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ in $\mathbb{R}^2$, and consider the standard basis for $\mathbb{R}^2$, $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$.

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{e}_1 + 6 \cdot \mathbf{e}_2$$

We say that 1 and 6 are the coordinates relative to the standard basis. However, this could be done for any basis of $\mathbb{R}^2$.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ form a basis for a vector space V. Every vector $\mathbf{x}$ in V is a linear combination of the elements of $\mathcal{B}$.

$$\mathbf{x} = c_1\boldsymbol{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

The weights c_1, c_2, …, c_n are called the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$).

These coordinates can be written $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Ex. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, and suppose there is some vector $\mathbf{x}$ such that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

From the earlier example, note that $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$.

Ex. Consider the basis $\mathcal{B} = \{1, t, t^2\}$ for $\mathbb{P}_2$. Find the $\mathcal{B}$-coordinates of $\mathbf{x} = (t + 3)^2$.

Ex. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the $\mathcal{B}$-coordinates of $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

In the previous example, we could form the matrix $P_{\mathcal{B}}$ whose columns are the vectors in $\mathcal{B}$:

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2]$$

Then solving the equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ becomes

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

We call $P_{\mathcal{B}}$ the change of coordinates matrix from $\mathcal{B}$ to the standard basis: $[\mathbf{x}]_{\mathcal{B}} \mapsto \mathbf{x}$

Since $\mathcal{B}$ is a basis, $P_{\mathcal{B}}$ is an invertible matrix, so

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}{}^{-1}\mathbf{x}$$

This maps the other direction: $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$

In the earlier example, we found the $\mathcal{B}$-coordinates of $\mathbf{x} = (t + 3)^2$ were $\begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix}$

This method of finding coordinates allows us to map an abstract vector space, such as $\mathbb{P}_2$, to the more concrete vector space $\mathbb{R}^3$.

→ This mapping is one-to-one and onto.

→ This mapping is linear:

$$[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + c_2[\mathbf{u}_2]_{\mathcal{B}} + \ldots + c_n[\mathbf{u}_n]_{\mathcal{B}}$$

Any one-to-one linear transformation from a vector space V onto a vector space W is called an isomorphism and the vector spaces are said to be isomorphic.

→ This means that they may look and feel completely different, but they act the same and are indistinguishable.

→ So $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$.

→ In general, any vector space whose basis has n elements is isomorphic to $\mathbb{R}^n$.

Ex. Show that $1 + 2t^2$, $4 + t + 5t^2$, and $3 + 2t$ are linearly dependent in $\mathbb{P}_2$.

Ex. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}, \mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

Then $\mathcal{B}$ is a basis for $H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x}$ is in H and, if it is, find the $\mathcal{B}$-coordinate for $\mathbf{x}$.

In the previous example, H represented a plane in $\mathbb{R}^3$.

→ We've just shown that this subspace of $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

Dimensions of a Vector Space

If a basis for a vector space V contains n vectors, we say that V is finite-dimensional and the dimension of V, written dim V, is n.

- The dimension of the zero subspace $\{\mathbf{0}\}$ is defined as 0.
- If V is not spanned by a finite set, we say V is infinite-dimensional.
- If dim $V = n$, then V is isomorphic with $\mathbb{R}^n$.

Ex. The dimension of $\mathbb{P}_2$ is 3 because its basis is $\{1, t, t^2\}$.

Ex. $\mathbb{P}$ is infinite-dimensional.

Ex. Find the dimension for $H = \text{span}\left\{\begin{bmatrix}3\\6\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$.

Ex. Find the dimension for $H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} \right\}$.

Ex. Find the dimension for $K = \text{span}\{2t^2 + 2, (t + 1)^2, t\}$.

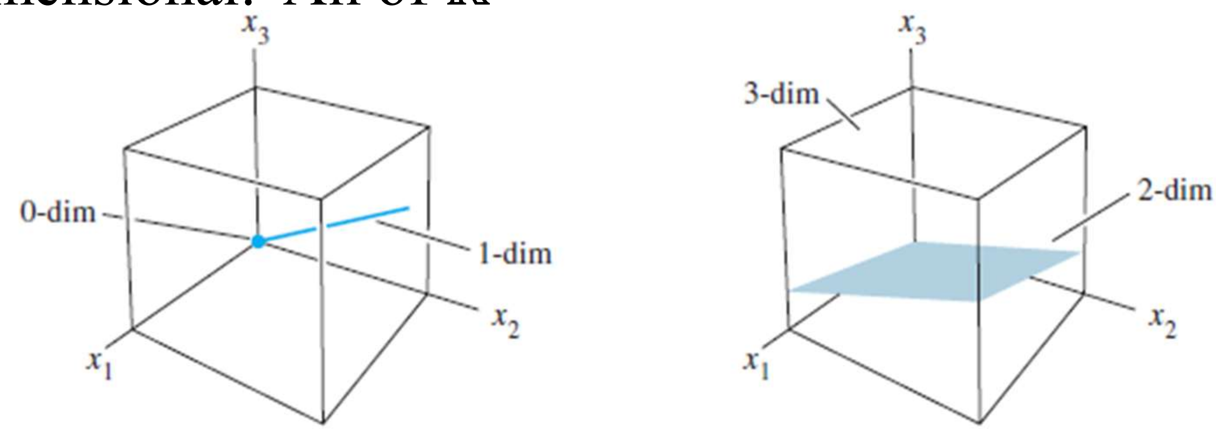
Consider the subspaces of $\mathbb{R}^3$

0-dimensional: Only the zero subspace

1-dimensional: Multiples of a single vector, so lines through the origin

2-dimensional: Linear combinations of two independent vectors, so planes through the origin

3-dimensional: All of $\mathbb{R}^3$



Going back to Nul A and Col A

The dimension of Nul A is the number of free variables of the equation $A\mathbf{x} = \mathbf{0}$.

The dimension of Col A is the number of pivot columns of A.

<u>Ex.</u> Find the dimensions of Nul A and Col A.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Before we can talk about rank, we need to define the <u>row space</u> of A, denoted Row A, and the subspace that is the span of the rows of A.

<u>Ex.</u> The row space of $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$ is a subspace of $\mathbb{R}^5$, and we can write the vectors horizontally if we wish.

<u>Note:</u> Row A = Col A^T

Thm. If two matrices A and B are row equivalent, their row spaces are the same. In addition, if B is in echelon form, its nonzero rows form a basis for the row space of A as well as B.

- This works because row operations that would result in B are just linear combinations of the rows of A

Ex. Find the bases for Row A, Col A, and Nul A for $A =$

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

Some observations:

- The basis for Col A used entries of A, but the bases for Nul A and Row A had no connection to the entries of A.
- Although the first 3 rows of the echelon form are independent, we can't assume the same is true of A.

Def. The rank of a matrix is the dimension of Col A.

Thm. Rank Theorem

- Col A and Row A have the same dimensions
- If A is an $m \times n$ matrix, Rank A + dim (Nul A) = n

Why are these true?

Ex. If A is 7×9 with a two-dimensional null space, what is the rank of A?

Ex. Could a 6×9 matrix have a two-dimensional null space?

Ex. Suppose a homogeneous system of equations with 18 equations and 20 variables if found to have a two-dimensional set of solutions. Does every associated non-homogeneous system have a solution?

Thm. Invertible Matrix Theorem

Let A be $n \times n$. The following are equivalent:

i. A is invertible

ii. A is row equivalent to I.

iii. A has n pivot positions (one in each row and column).

iv. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

v. The columns of A are linearly independent.

vi. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

vii. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.

viii. The columns of A span $\mathbb{R}^n$.

ix. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

x. The determinant of A is not zero

xi. Col $A = \mathbb{R}^n$

xii. Row $A = \mathbb{R}^n$

xiii. Nul A is the zero subspace