Inner Product

The inner product of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

is defined as

$$\mathbf{u}^T \cdot \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is the dot product between **u** and **v**.

Ex. Compute $\mathbf{u} \cdot \mathbf{v}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

Properties of the Inner Product

1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

2) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

3) $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

4) $\mathbf{a} \cdot \mathbf{a} \geq 0$ and $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$

Def. The length (or norm) of a vector is denoted $||\mathbf{a}||$

Thm. The length of vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$

The length of vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Note 1: $||\mathbf{a}||^2 = \mathbf{a} \cdot \mathbf{a}$

Note 2: $||c\mathbf{a}|| = |c|\ ||\mathbf{a}||$ ← Prove it.

Def. A unit vector is a vector whose length is 1.

→ This process is sometimes called normalization.

The unit vector in the direction of **a** is

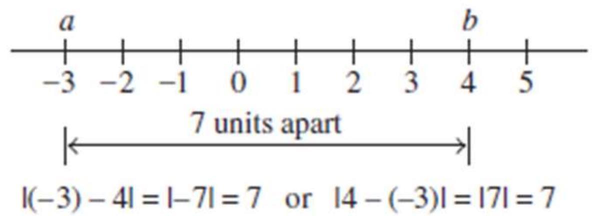
$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

Ex. Find the unit vector in the direction of $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \end{bmatrix}$

Ex. Let W be a subspace of $\mathbb{R}^2$ spanned by $\mathbf{v} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$.
Find a unit vector that is a basis for W.

Distance:

In 1-D, the distance between points a and b is $|b - a|$



In higher dimensions, this can be extended to be

$$\text{dist}(\mathbf{u},\mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

Ex. Find $\text{dist}\left(\begin{bmatrix} 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$.

Note this is same as using the distance formula on the points (7,1) and (3,2).

The inner product is used to find the angle between two vectors:

Thm. If θ is the angle between **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

Ex. Find the angle between $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

Thm. Two vectors **a** and **b** are orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$.

Orthogonal = Perpendicular = Normal

Ex. Show that $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$ are orthogonal

Thm. Pythagorean Theorem

Two vectors **u** and **v** are orthogonal if and only if

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

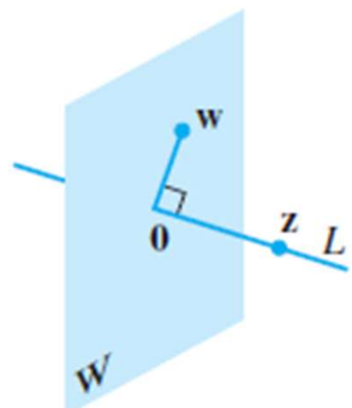
Ex. Verify this for $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$

Def. If $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, we say $\mathbf{z}$ is orthogonal to W.

The set of all vectors orthogonal to W is called the orthogonal complement of W, and is denoted $W^{\perp}$ (called "W perpendicular" or "W perp").

- $W^{\perp}$ is itself a subspace of $\mathbb{R}^n$.

If W is a plane (through the origin) in $\mathbb{R}^3$, then $W^{\perp}$ is the set of all vectors orthogonal to the plane.



- A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.

<u>Ex.</u> Is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $W^{\perp}$ where $W = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}\right\}$?

Ex. Evaluate $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

<u>Note:</u> $$A\mathbf{x} = \begin{bmatrix} (Row_1 A) \cdot \mathbf{x} \\ (Row_2 A) \cdot \mathbf{x} \\ \vdots \\ (Row_n A) \cdot \mathbf{x} \end{bmatrix}$$

→ If $A\mathbf{x} = \mathbf{0}$ (meaning $\mathbf{x}$ is in Nul A) then $(\text{Row}_k A) \cdot \mathbf{x} = 0$

→ $\mathbf{x}$ is orthogonal to every row of A

→ $\mathbf{x}$ is in $(\text{Row } A)^\perp$

$$(\text{Row } A)^\perp = \text{Nul } A$$

Also, Col A = Row (A^T)

→ $(\text{Col } A)^\perp = (\text{Row } A^T)^\perp = \text{Nul } A^T$

Def. A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ of vectors in $\mathbb{R}^n$ is called an orthogonal set if each pair of distinct vectors in the set is orthogonal.

→ $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ as long as $i \neq j$

Ex. Show that $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$ is an orthogonal set.

An orthogonal set is linearly independent.

→ Let’s prove it.

Def. An orthogonal basis is a basis for a subspace that is also an orthogonal set.

→ The standard basis of $\mathbb{R}^n$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis.

→ An orthogonal basis is useful because it's easy to compute the weights.

Thm. If $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_p}\}$ is an orthogonal basis of W, and if $\mathbf{y}$ is in W ($\mathbf{y} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_p\mathbf{b}_p$), then the weights are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{b}_j}{\mathbf{b}_j \cdot \mathbf{b}_j}$$

→ This makes it easy to find the coordinate vector

$$[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

→ Let's prove it.

Ex. Express $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in the orthogonal basis $S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$.

Without this trick, we have to solve the system of equations.

Given a vector **u** in $\mathbb{R}^n$, consider the problem of decomposing a vector **y** into the sum of two vectors: one that is in the direction of **u** and the other that is orthogonal to **u**:

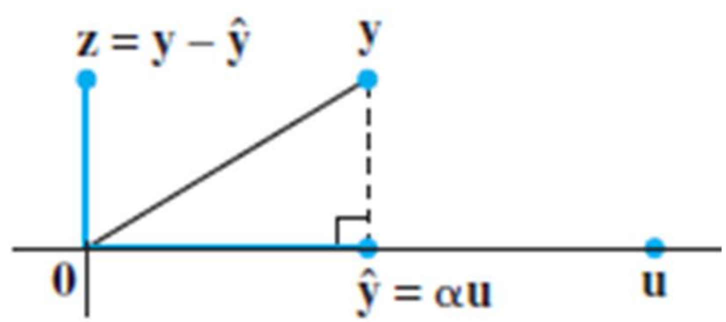
$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

$\hat{\mathbf{y}}$ is in the direction of **u**, so $\hat{\mathbf{y}} = \alpha\mathbf{u}$.

→ This is called the orthogonal projection of **y** onto **u**

z is called the component of **y** orthogonal to **u**

→ Let's find a formula for α

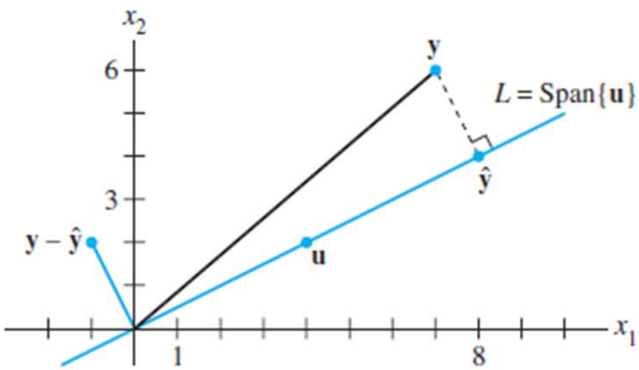


If the L is the line containing $\mathbf{u}$ (all multiples of $\mathbf{u}$), we can write

$$\hat{\mathbf{y}} = \mathrm{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

Ex. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

Ex. In the previous example, find the distance from **y** to the line containing **u**.



Def. An orthonormal set is an orthogonal set of unit vectors.

If a subspace is spanned by an orthonormal set, then the set is called an orthonormal basis for the subspace.

Ex. Show that $\left\{ \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix} \right\}$ is an orthonormal set.

This is the earlier orthogonal set where each vector is made into a unit vector.

Thm. A matrix U (not necessarily square) has orthonormal columns if and only if $U^{\mathrm{T}}U = I$.

→ Prove it

Thm. Let U be a matrix with orthonormal columns.

i. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

ii. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

iii. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

→ This means that the transformation $\mathbf{x} \mapsto U\mathbf{x}$ preserves length and orthogonality

If U is a square matrix with orthonormal columns, then

$$U^T U = I$$

→ $U^T = U^{-1}$

→ We call this an <u>orthogonal matrix</u>

→ In this case, the rows of U are orthonormal as well

$$\begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$$