Orthogonal Projections

Last class, we projected a vector **y** onto a line that was the span of a vector **u** → a subspace with dimension 1

Today, we will discuss projecting a vector **y** onto a subspace that has a dimension greater than 1

Consider $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$, an orthogonal basis of $\mathbb{R}^5$ and the vector $\mathbf{y}$ in $\mathbb{R}^5$.

Consider the subspace $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5$$

$$\mathbf{y} = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2) + (c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5)$$

$$\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$$

→ $\mathbf{z}_1$ is in W, let's show $\mathbf{z}_2$ is in $W^{\perp}$

This means that $W^{\perp} = \text{span}\{\mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$

Thm. Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Every vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

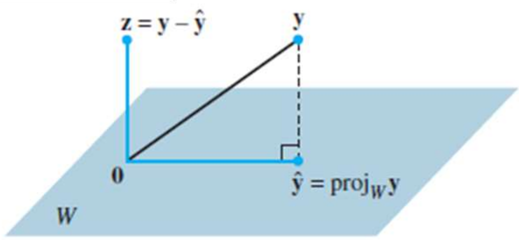
where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

$\hat{\mathbf{y}}$ is called the orthogonal projection of $\mathbf{y}$ onto W and is sometimes written $\text{proj}_W \mathbf{y}$.



<u>Ex.</u> Let $\left\{\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$ be an orthogonal basis for W. Write $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as the sum of a vector in W and a vector in $W^{\perp}$.

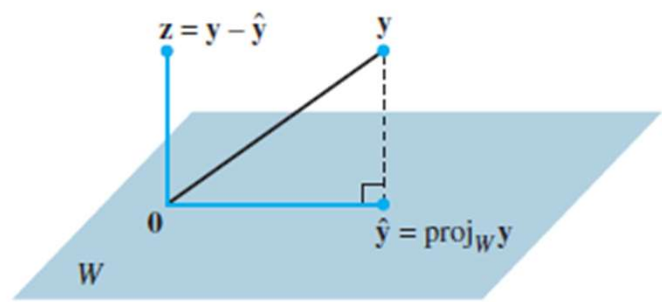
Thm. Best Approximation Theorem

$\hat{\mathbf{y}}$ is the vector in W that is closest to $\mathbf{y}$, in the sense that, for any vector $\mathbf{v}$ in W,

$$\|\mathbf{y} - \hat{\mathbf{y}}\| \le \|\mathbf{y} - \mathbf{v}\|$$

$\hat{\mathbf{y}}$ is called the best approximation of $\mathbf{y}$ by elements of W.

Because we haven't discussed the basis of W, this means that $\hat{\mathbf{y}}$ is the same no matter what basis is used for W.



Ex. Find the distance from $\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$ to $W = \text{span}\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\cancel{\mathbf{u}_1 \cdot \mathbf{u}_1}} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\cancel{\mathbf{u}_2 \cdot \mathbf{u}_2}} \mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\cancel{\mathbf{u}_p \cdot \mathbf{u}_p}} \mathbf{u}_p$$

If we define $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_p]$, then $\hat{\mathbf{y}} = UU^T\mathbf{y}$

→ Prove it

Ex. Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ (note its orthogonal).

Find a matrix A such that $\text{proj}_W \mathbf{y} = A\mathbf{y}$ for any vector $\mathbf{y}$.

We've seen that it's useful to have an orthogonal basis

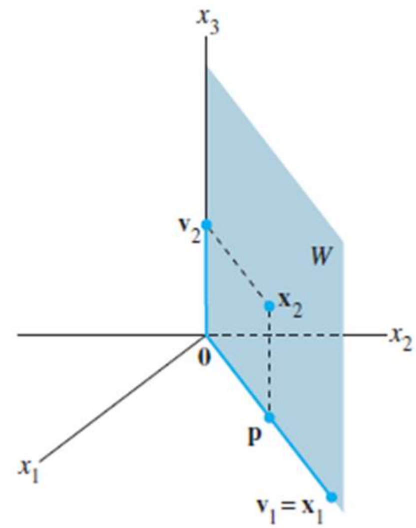
→ If we are given some other basis, we can find an orthogonal basis using the Gram-Schmidt Process

$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ a basis of W → $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ an orthogonal basis of W

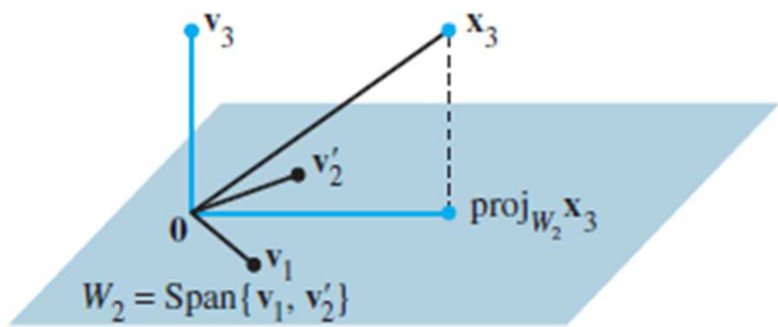
Gram-Schmidt Process

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\text{span}\{\mathbf{v}_1\}}\mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$



$$\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}}\mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$



Ex. Let $W = \text{span}\left\{\begin{bmatrix}3\\6\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$, find an orthogonal basis.

Ex. Let $W = \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$, find an orthogonal basis.

Ex. Let $W = \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$, find an orthonormal basis.

Ex. Let $W = \text{Nul}\begin{bmatrix} 1 & -1 & -2 \\ 2 & -2 & -4 \\ 4 & -4 & -8 \end{bmatrix}$, find the point in W that is closest to $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and find the distance from $\mathbf{y}$ to W.