Inner Product Spaces

All of the inner products that we've computed, and all of the results, have been in $\mathbb{R}^n$.

→ For other spaces, we will define the inner product to have the same properties as it has in $\mathbb{R}^n$.

Def. An inner product on a vector space V is a function that satisfies the following axioms for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and for all scalars c.

i. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

ii. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

iii. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

iv. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

A vector space with an inner product is called an inner product space.

→ The dot product is considered the standard inner product

→ Nearly everything we've discussed for $\mathbb{R}^n$ in this chapter carries over to other inner product spaces

Ex. Consider the vector space $\mathbb{P}_2$ with inner product $\langle p, q\rangle = p(0)q(0) + p\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + p(1)q(1)$. If $p(t) = 12t^2$ and $q(t) = 2t - 1$, compute $\langle p, q\rangle$ and $\langle q, q\rangle$.

Ex. For any two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, define $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1 v_1 + 5u_2 v_2$. Determine if this is an inner product.

This is sometimes called a weighted dot product.

Ex. For any two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, define $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 - 5u_2v_2$. Determine if this is an inner product.

Weighted dot products only work if the weights are positive.

Ex. Consider the vector space $C[a,b]$ and define $\langle f(t), g(t)\rangle = \int_a^b f(t)g(t)dt$. Determine if this is an inner product.

We can extend concepts we used with dot product to more general inner products:

- The lengths of a vector is $||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle$
- A unit vector is a vector with length 1
- The distance between **u** and **v** is $||\mathbf{u} - \mathbf{v}||$
- Two vectors are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

Ex. Consider the vector space $\mathbb{P}_2$ with inner product $\langle p, q \rangle = p(0)q(0) + p\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + p(1)q(1)$. If $p(t) = 12t^2$ and $q(t) = 2t - 1$, find the length of p and q.

Ex. Consider the vector space $C[a,b]$ and define $\langle f(t), g(t)\rangle = \int_a^b f(t)g(t)dt$. Find equations to find the length of f and the distance between f and g.

Ex. Consider the vector space $C[-\pi, \pi]$ and define $\langle f(t), g(t) \rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$. Show that $f(x) = \sin x$ and $g(x) = \cos x$ are orthogonal.

In the next example, we'll extend what we've learned about dot product to our function inner product space.

If f is in C and W is a subspace with orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, then

$$\text{proj}_W f = \frac{\langle f, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 + \frac{\langle f, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2$$

$$\text{dist}(f, W) = \|f - \text{proj}_W f\|$$

To find the orthogonal basis, we'll use the Gram-Schmidt process.

Ex. Consider the vector space $C[0,1]$ and functions $f_1(t) = 1, f_2(t) = t$, and $f_3(t) = t^2$, and define $\langle f(t), g(t) \rangle = \int_0^1 f(t)g(t)dt$. Find the distance from f_3 to span$\{f_1, f_2\}$.

Ex. Diagonalize $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

Not all matrices can be diagonalized. This section finds criteria that identify matrices that can.

Certainly, if an $n \times n$ matrix has n distinct eigenvalues, then it can be diagonalized.

Def. A matrix is symmetric if $A^{\mathrm{T}} = A$, meaning $a_{ij} = a_{ji}$.

Symmetric:

$$\begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & -8 \\ 3 & 1 & 4 \\ -8 & 4 & 7 \end{bmatrix}$$

Not Symmetric:

$$\begin{bmatrix} 1 & 5 \\ -2 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 7 & -8 \\ 3 & 1 & 5 \\ -8 & 4 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & -9 \\ 5 & 8 & 2 \end{bmatrix}$$

Remember: An orthogonal matrix P is a matrix in which the columns are orthonormal. Additionally, $P^{-1} = P^{T}$.

Def. A matrix is orthogonally diagonalizable if there is an orthogonal matrix P and diagonal matrix D such that
$A = PDP^{-1}$

Thm. A matrix is orthogonally diagonalizable if and only if it is symmetric.

→ If a matrix is symmetric, it is automatically orthogonally diagonalizable.

→ Some non-symmetric matrices can still be diagonalized, but there's no quick way to check this.

Ex. Orthogonally diagonalize $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. The characteristic equation is $0 = -(\lambda + 2)(\lambda - 7)^2$.