Math 200 – Linear Algebra

Andy Rosen

www.rosenmath.com

Systems of Linear Equations

A system of linear equations (or linear system) refers to multiple equations involving multiple variables

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad\qquad - 4x_3 = -7 \end{cases}$$

Linear means each variable is to the 1st power

Coefficients are the numbers multiplying the variables

In general, a linear equation would look like

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad\qquad - 4x_3 = -7 \end{cases}$$

A solution is values of the variables that satisfy both equations.

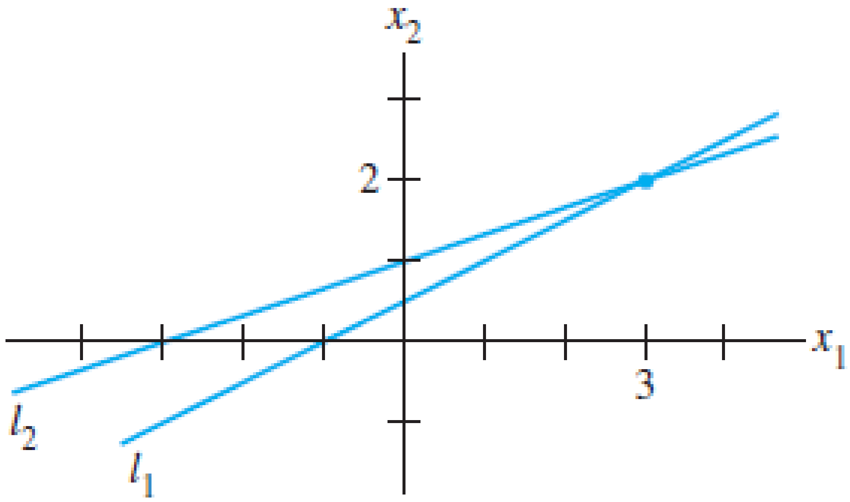
→ (5, 6.5, 3) is a solution to the system above

→ There are other solutions to this system. The collection of all possible solutions to a system is called the solution set

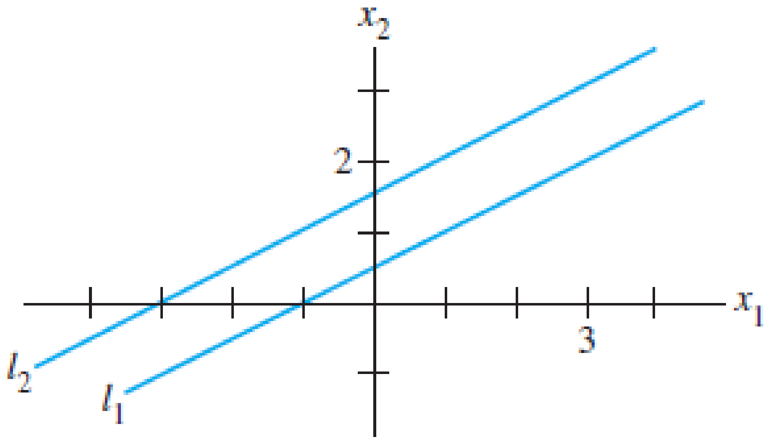
Two systems are equivalent if they have to same solution set.

When solving a linear system with 2 variables and 2 equations, the solution is the point where the graphs of the lines intersect:

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

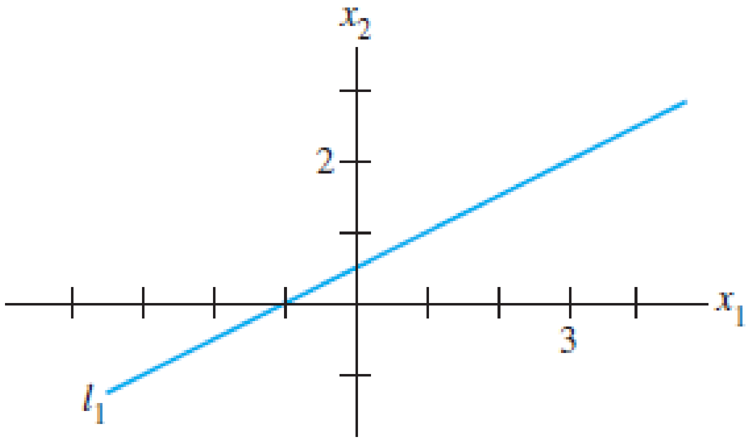


$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases}$$



These lines are parallel
→ No solution
→ The system is inconsistent

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$$



These lines coincide
→ Infinitely many solution

A matrix is a rectangular array that can help us to streamline the solving of a system of equations

$$\begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & 9 \end{bmatrix}$$

The size of this matrix is 2×3

By changing a system of equations into a matrix (augmented matrix), we can make it easier to work with

$$\begin{cases} x_1 - 4x_2 + 3x_3 = 5 \\ -x_1 + 3x_2 - x_3 = -3 \\ 2x_1 \qquad + 4x_3 = 6 \end{cases} \Rightarrow \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{bmatrix}$$

The left side is called the coefficient matrix

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & 4 \end{bmatrix}$$

The following row operations will produce an equivalent system:

- Rows can be switched (interchange)

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 & -1 & -3 \\ 1 & -4 & 3 & 5 \\ 2 & 0 & 4 & 6 \end{bmatrix}$$

The following row operations will produce an equivalent system :

• A row can be multiplied by a non-zero constant (scaling)

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 8 & 0 & 16 & 24 \end{bmatrix}$$

The following row operations will produce an equivalent system :

• A row can be replaced by the sum of itself and a multiple of another row (replacement)

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 1 & 3 & 3 & 3 \end{bmatrix}$$

We are going to use row operations to put a matrix into echelon form

- Any row with all zeroes is at the bottom
- Each lead entry (nonzero entry) has zeroes below it

If each lead entry is 1, and if each 1 has zeroes above and below it, we say the matrix is in reduced echelon form

These are in echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 3 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

This is also called <u>triangular form</u>.

These are in reduced echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

To put a matrix into reduced echelon form:

- Use scaling or interchange to place a non-zero number (preferably 1) in the upper-left entry, then use it with replacement to get zeroes below
- Move down and right, repeating the process to put the matrix in echelon form
- Starting with the bottom right 1, work backward to put the matrix in reduced echelon form

Ex. Solve the system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

Because there was a solution, the system is consistent.

A matrix in echelon form may be different depending on the row operations you choose

→ The reduced echelon form will be unique

Another option is to leave the augmented matrix in echelon form and then backsolve

Ex. Solve the system

$$\begin{cases} \quad\quad x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

Because there was no solution, the system is inconsistent.

A <u>pivot position</u> in matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A <u>pivot column</u> is a column of A that contains a pivot position.

→ The previous system was inconsistent because the rightmost column was a pivot column

Ex. Reduce the matrix to echelon form and locate the pivot columns

$$\begin{bmatrix} 0 & -3 & -6 & 4 & -9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Consider the system whose augmented matrix has been changed into the reduced echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 \quad -5x_3 = 1 \\ \quad x_2 + x_3 = 4 \\ \qquad 0 = 0 \end{cases}$$

The variables x_1 and x_2 are called basic variables. The variable x_3 is called a free variable because it is free to be anything.

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 = \text{free} \end{cases}$$

Choosing different values of x_3 results in different solutions

Ex. Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

This is called a parametric description of the solution set.

→ The free variables can be considered to be parameters

Note that the basic variables corresponded with the pivot columns.

If a system is consistent, then the solution set contains either

i. A unique solution, so there are no free variables, or
ii. Infinitely many solutions, so there is at least one free variable.