Vectors

A matrix with only one column is called a column vector, or simply a vector.

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The set of all vectors with 2 entries is $\mathbb{R}^2$ (read R-two), since each of the two entries can be any real number.

Two vectors are equal if the corresponding entries are equal.

Ordered pairs in the *xy*-plane, like vectors in $\mathbb{R}^2$, are represented by two numbers.

We can identify the plotted point (3,-1) with the column vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Sometimes, it is useful to include a directed line segment (arrow) from the origin to the point, though we are not interested in any of the points on the segment.

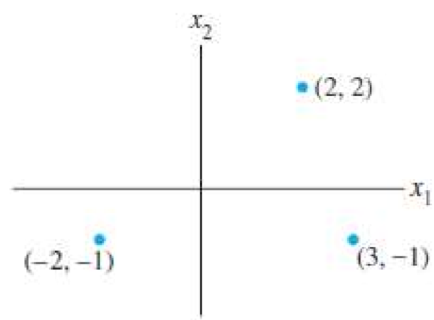


FIGURE 1 Vectors as points.

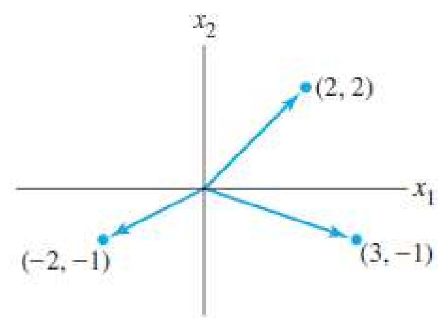


FIGURE 2 Vectors with arrows.

Adding and subtracting vectors means performing the operations on corresponding entries

<u>Scalar multiplication</u> means multiplying a vector by a constant (scalar)

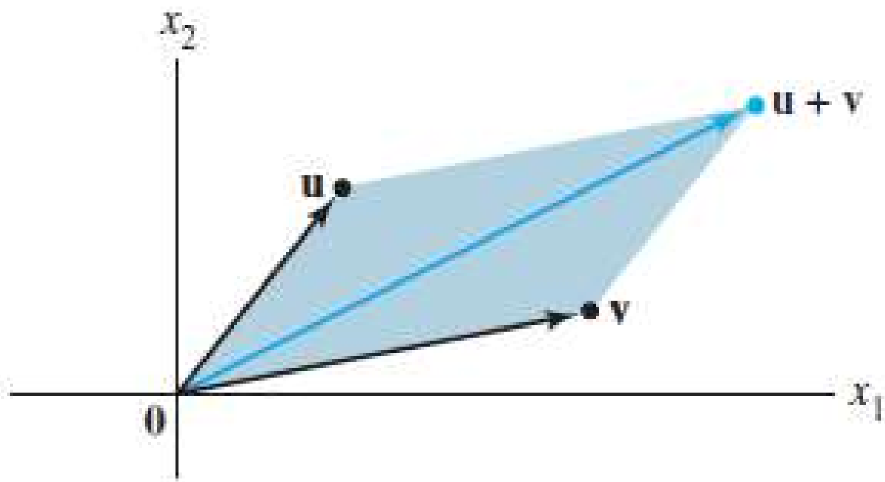
→ We do this by multiplying each entry by the constant

Ex. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

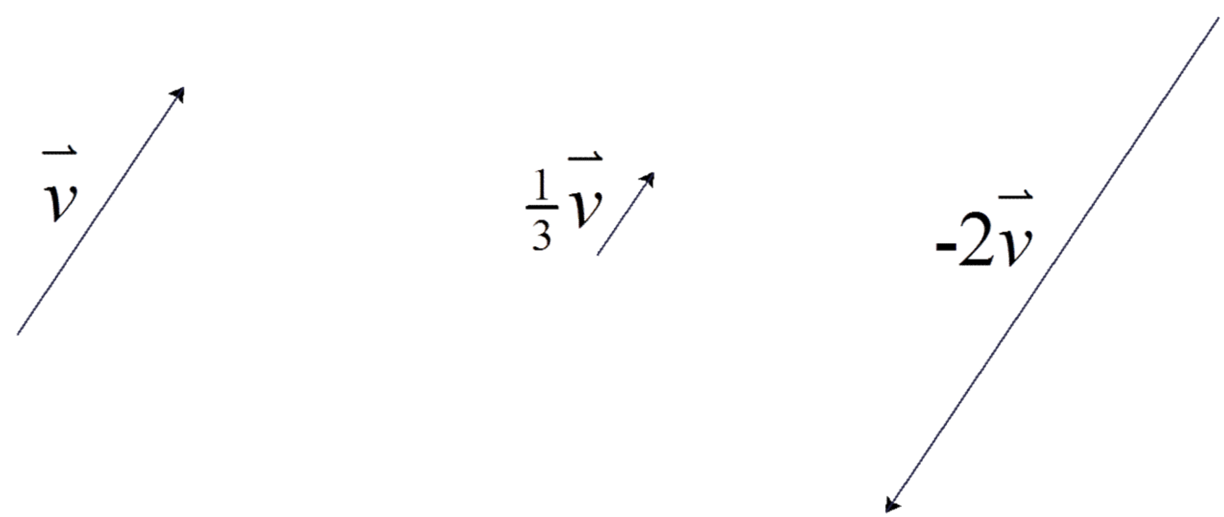
a. $3\mathbf{u}$

b. $3\mathbf{u} - \mathbf{v}$

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the xy-plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.



Def. If c is a scalar and **v** is a vector, then c**v** is the vector with the same direction as **v** that has length c times as long as **v**. If $c < 0$, then c**v** goes in the opposite direction as **v**.



These ideas can be extended to n-dimensional space, $\mathbb{R}^n$.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The zero vector, $\mathbf{0}$, is the vector whose entries are all zero.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

A linear combination of vectors involves multiplying each vector by a constant coefficient and adding the results.

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

→ The vector $\mathbf{u} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$ is a linear combination of

$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ because $\mathbf{u} = 3\mathbf{v}_1 + 2\mathbf{v}_2$.

The coefficients are called the weights of the combination

Ex. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Notice that the columns of our augmented matrix were $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$.

→ We can abbreviate by writing $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}]$

<u>In general:</u>

A <u>vector equation</u> $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \quad \mathbf{b}]$

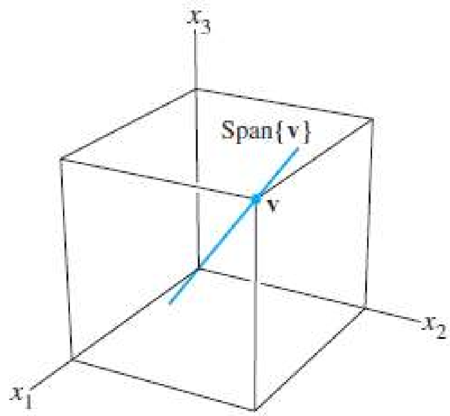
Ex. Convert $\begin{cases} 3x_1 - 2x_2 + x_3 = 4 \\ -x_1 + 5x_2 + 2x_3 = 6 \\ 2x_1 - x_2 - 5x_3 = 2 \end{cases}$ to a vector equation.

Def. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

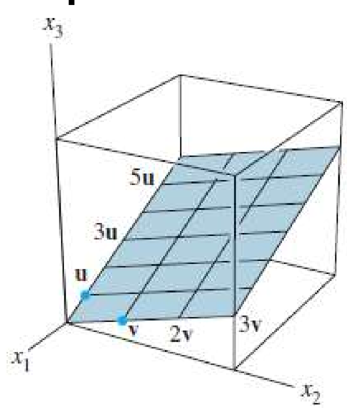
That is, $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the set of all vectors that can be written $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$, where $c_1,\ldots,c_p$ are scalars.

In $\mathbb{R}^3$:

Span{**v**} is the line through the origin and **v**:



Span{**u**,**v**} is the plane through the origin, **u** and **v**:



Ex. Determine if **b** is in the plane generated by Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

The Matrix Equation

Let A be the matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n]$, where each of the $\mathbf{a}$'s is a vector in $\mathbb{R}^m$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product $A\mathbf{x}$ is the linear combination of the columns of A using the entries of $\mathbf{x}$ as weights:

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Ex. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

Ex. $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

Ex. $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

Linear system:
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

Vector equation:
$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Matrix Equation:
$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Linear systems can be expressed in 3 different ways, we can pick the one that's most convenient.

Ex. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, and b_3?

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Thm. Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$. The following are equivalent (all are true or none are true):

i. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ in $\mathbb{R}^m$.

ii. Every $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A

iii. The columns of A span $\mathbb{R}^m$ (every vector in $\mathbb{R}^m$ is in the span of the columns of A)

iv. A has a pivot position in every row

Note: This is about the coefficient matrix, A, of a linear system, not the augmented matrix $[A \quad \mathbf{b}]$.

Ex. Can $A\mathbf{x} = \mathbf{b}$ be solved for any $\mathbf{b}$ in $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 0 & -1 & 6 \\ 7 & 1 & -1 & 14 \\ 5 & 1 & 1 & 2 \end{bmatrix}$$

Ex. Do the columns of A span $\mathbb{R}^3$?

$$A = \begin{bmatrix} 7 & 1 & 2 \\ 5 & -1 & 6 \\ -2 & 0 & 4 \end{bmatrix}$$

Let's do these again using dot product:

Ex. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

Ex. $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

The identity matrix is a square matrix that has ones on its main diagonal and zeroes as every other entry

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying any vector by I results in the same vector

$$I\mathbf{x} = \mathbf{x}$$

Solution Sets of Linear Systems

The linear system $A\mathbf{x} = \mathbf{0}$ is called homogeneous.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

This system always has at least 1 solution, where all the x's are 0. This is called the trivial solution.

Thm. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

→ So the homogeneous system has either one trivial solution or infinitely many solutions.

Ex. Describe the solution set of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

1 free variable resulted in a line in $\mathbb{R}^3$.

Ex. Describe the solution set of $10x_1 - 3x_2 - 2x_3 = 0$

2 free variables resulted in a plane in $\mathbb{R}^3$.

If A has no free variables:

- Trivial solution
- The point $\mathbf{0}$ in $\mathbb{R}^3$

If A has 1 free variable:

- A line in $\mathbb{R}^3$ that passes through the origin
- Can be described parametrically by $\mathbf{x} = t\mathbf{v}_1$.

If A has 2 free variables:

- A plane in $\mathbb{R}^3$ that passes through the origin
- Can be described parametrically by $\mathbf{x} = s\mathbf{v}_1 + t\mathbf{v}_2$.

→ Note this represents Span$\{\mathbf{v}_1, \mathbf{v}_2\}$

When we write our solution sets in this form, it is called the parametric vector form.

If $\mathbf{b} \neq \mathbf{0}$, the linear system $A\mathbf{x} = \mathbf{b}$ is called <u>non-homogeneous</u>.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Ex. Describe the solution set of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

<u>$A\mathbf{x} = \mathbf{b}$ has no solutions if:</u>

- $A\mathbf{x} = \mathbf{b}$ is inconsistent

<u>$A\mathbf{x} = \mathbf{b}$ has 1 solution if:</u>

- The corresponding homogeneous system had only the trivial solution

<u>$A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if:</u>

- The corresponding homogeneous system had infinitely many solutions
- Solutions would be 1 vector plus a linear combination of vectors that satisfy the corresponding homogeneous system.
- $\mathbf{x} = \mathbf{p} + t\mathbf{v}_1 \rightarrow$ a line not through the origin
- $\mathbf{x} = \mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2 \rightarrow$ a plane not through the origin

Prove the previous result: