Change of Basis

Some problems are easier to work in a different basis.

→ We need to talk about how to change between bases.

Ex. Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ such that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$. Suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{C}}$.

We can notice that $[\mathbf{x}]_{\mathcal{C}}$ was equal to the product of a matrix and $[\mathbf{x}]_{\mathcal{B}}$.

Further, we notice that the columns of the matrix were the $\mathcal{C}$-coordinates of the vectors in $\mathcal{B}$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathrm{P}} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}}]$$

This is called the <u>change-of-coordinates</u> matrix from $\mathcal{B}$ to $\mathcal{C}$

→ To change from $\mathcal{C}$ to $\mathcal{B}$, we can use the inverse

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathrm{P}} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathrm{P}}\right)^{-1}$$

In general, we can find the change-of-coordinates matrix by using

$$[P_{\mathcal{B}} \quad P_{\mathcal{C}}] \sim [I \quad \underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathrm{P}}]$$

Ex. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ be the vectors of bases $\mathcal{B}$ and $\mathcal{C}$. Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathrm{P}}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathrm{P}}$

Eigenvalues and Eigenvectors

Def. Let A be a square matrix. A number λ is an eigenvalue of A if there is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

$\mathbf{x}$ is called an eigenvector corresponding to λ.

Ex. Show that $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$

Ex. Show that $\lambda = 7$ is an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

In the last example, we solved the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$

→ This is the null space of the matrix $A - \lambda I$

→ This is a subspace which we call the eigenspace of A corresponding to λ.

Ex. An eigenvalue of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ is $\lambda = 2$. Find a basis for the corresponding eigenspace.

Thm. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent.

→ This fact will be useful later.

Ex. Show that $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

The eigenvalues of a triangular matrix are the entries on the main diagonal.

→ Note that $\lambda = 0$ is also an eigenvalue in the previous example

→ When 0 is an eigenvalue, this means that $A\mathbf{x} = 0\mathbf{x}$, or $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

→ This means that A is not invertible

Thm. Invertible Matrix Theorem

Let A be $n \times n$. The following are equivalent:

i. A is invertible
ii. A is row equivalent to I.
iii. A has n pivot positions (one in each row and column).
iv. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
v. The columns of A are linearly independent.
vi. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
vii. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.
viii. The columns of A span $\mathbb{R}^n$.
ix. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
x. The determinant of A is not zero
xi. Col $A = \mathbb{R}^n$
xii. Row $A = \mathbb{R}^n$
xiii. Nul A is the zero subspace
xiv. The number 0 is not an eigenvalue of A

For more complicated matrices A, we need a process to find eigenvalues and eigenvectors:

→ $(A - \lambda I)\mathbf{x} = 0$

→ By the Invertible Matrix Theorem, this has a nontrivial solution only when $(A - \lambda I)$ is not invertible

→ This means that $\det(A - \lambda I) = 0$

→ The equation $\det(A - \lambda I) = 0$ is called the <u>characteristic equation</u> of A.

Ex. Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Ex. Find the eigenvalues and bases for the corresponding eigenspaces of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

When A is $n \times n$, the characteristic equation is degree n and is much harder to solve. To make life easier, let's review some properties of determinants:

- Let U be an echelon form of A obtained by replacements and interchanges (no scaling), and let r be the number of interchanges, then

$$\det A = \begin{cases} (-1)^r \det U & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

When A is $n \times n$, the characteristic equation is degree n and is much harder to solve. To make life easier, let's review some properties of determinants:

- det AB = (det A)(det B)
- det (A^T) = det A
- If A is triangular, det A is the product of the entries on the main diagonal
- Row replacement doesn't change the determinant
- Row interchange changes the sign of the determinant
- Row scaling also scales the determinant by the same factor

Ex. Find the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

Ex. Find the characteristic equation of $A = \begin{bmatrix} 3 & 6 & -8 & 4 \\ 0 & 1 & 6 & -6 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

We say that $\lambda = 3$ is an eigenvalue of multiplicity 2.

For this class, we will only focus on real valued eigenvalues.

- It should be noted, though, that eigenvalues are solutions to n^{th} degree polynomial equations, and so they can be complex

Also, in this class we will only be solving characteristic equations that are quadratic.

Def. Two matrices A and B are similar if there is some matrix P such that $A = PBP^{-1}$

We say that the transformation $A \mapsto PAP^{-1}$ is called a similarity transformation.

Thm. If two matrices are similar, then they have the same characteristic equation and, therefore, the same eigenvalues with the same multiplicities.

→ Let's prove it.