1. The auxiliary equation is $m^2 - m - 2 = (m+1)(m-2) = 0$ so that $y = c_1 x^{-1} + c_2 x^2$.

3. The auxiliary equation is $m^2 = 0$ so that $y = c_1 + c_2 \ln x$.

5. The auxiliary equation is $m^2 + 4 = 0$ so that $y = c_1 \cos(2\ln x) + c_2 \sin(2\ln x)$.

7. The auxiliary equation is $m^2 - 4m - 2 = 0$ so that $y = c_1 x^{2-\sqrt{6}} + c_2 x^{2+\sqrt{6}}$.

9. The auxiliary equation is $25m^2 + 1 = 0$ so that $y = c_1 \cos\left(\frac{1}{5}\ln x\right) + c_2 \sin\left(\frac{1}{5}\ln x\right)$.

11. The auxiliary equation is $m^2 + 4m + 4 = (m+2)^2 = 0$ so that $y = c_1 x^{-2} + c_2 x^{-2}\ln x$.

13. The auxiliary equation is $3m^2 + 3m + 1 = 0$ so that

$$y = x^{-1/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{6}\ln x\right) + c_2 \sin\left(\frac{\sqrt{3}}{6}\ln x\right)\right].$$

15. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2) - 6 = m^3 - 3m^2 + 2m - 6 = (m-3)(m^2+2) = 0.$$

Thus

$$y = c_1 x^3 + c_2 \cos\left(\sqrt{2}\ln x\right) + c_3 \sin\left(\sqrt{2}\ln x\right).$$

17. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) = m^4 - 7m^2 + 6m = m(m-1)(m-2)(m+3) = 0.$$

Thus

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^{-3}.$$

19. The auxiliary equation is $m^2 - 5m = m(m-5) = 0$ so that $y_c = c_1 + c_2 x^5$ and

$$W(1, x^5) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4.$$

Identifying $f(x) = x^3$ we obtain $u_1' = -\frac{1}{5}x^4$ and $u_2' = 1/5x$. Then $u_1 = -\frac{1}{25}x^5$, $u_2 = \frac{1}{5}\ln x$, and

$$y = c_1 + c_2 x^5 - \frac{1}{25}x^5 + \frac{1}{5}x^5 \ln x = c_1 + c_3 x^5 + \frac{1}{5}x^5 \ln x.$$

21. The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$ so that $y_c = c_1 x + c_2 x \ln x$ and

$$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x.$$

Identifying $f(x) = 2/x$ we obtain $u_1' = -2\ln x/x$ and $u_2' = 2/x$. Then $u_1 = -(\ln x)^2$, $u_2 = 2\ln x$, and

$$\begin{aligned} y &= c_1 x + c_2 x \ln x - x(\ln x)^2 + 2x(\ln x)^2 \\ &= c_1 x + c_2 x \ln x + x(\ln x)^2, \qquad x > 0. \end{aligned}$$

23. The auxiliary equation $m(m-1) + m - 1 = m^2 - 1 = 0$ has roots $m_1 = -1$, $m_2 = 1$, so $y_c = c_1 x^{-1} + c_2 x$. With $y_1 = x^{-1}$, $y_2 = x$, and the identification $f(x) = \ln x/x^2$, we get

$$W = 2x^{-1}, \qquad W_1 = -\ln x/x, \qquad \text{and} \qquad W_2 = \ln x/x^3.$$

Then $u_1' = W_1/W = -(\ln x)/2$, $u_2' = W_2/W = (\ln x)/2x^2$, and integration by parts gives

$$\begin{aligned} u_1 &= \frac{1}{2}x - \frac{1}{2}x\ln x \\ u_2 &= -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}, \end{aligned}$$

so

$$y_p = u_1 y_1 + u_2 y_2 = \left(\frac{1}{2}x - \frac{1}{2}x\ln x\right)x^{-1} + \left(-\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}\right)x = -\ln x$$

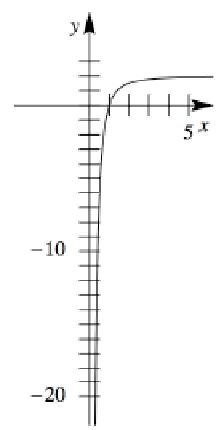
and

$$y = y_c + y_p = c_1 x^{-1} + c_2 x - \ln x, \qquad x > 0.$$

25. The auxiliary equation is $m^2 + 2m = m(m+2) = 0$, so that $y = c_1 + c_2 x^{-2}$ and $y' = -2c_2 x^{-3}$. The initial conditions imply

$$\begin{aligned} c_1 + c_2 &= 0 \\ -2c_2 &= 4. \end{aligned}$$

Thus, $c_1 = 2$, $c_2 = -2$, and $y = 2 - 2x^{-2}$. The graph is given to the right.



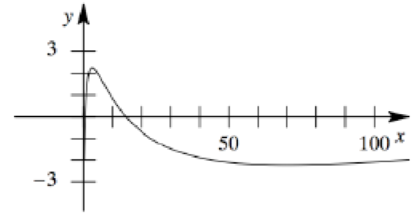
27. The auxiliary equation is $m^2 + 1 = 0$, so that

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

and

$$y' = -c_1 \frac{1}{x} \sin(\ln x) + c_2 \frac{1}{x} \cos(\ln x).$$

The initial conditions imply $c_1 = 1$ and $c_2 = 2$. Thus $y = \cos(\ln x) + 2\sin(\ln x)$. The graph is given to the right.



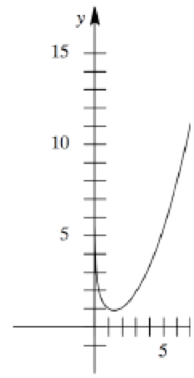
29. The auxiliary equation is $m^2 = 0$ so that $y_c = c_1 + c_2 \ln x$ and

$$W(1, \ln x) = \begin{vmatrix} 1 & \ln x \\ 0 & 1/x \end{vmatrix} = \frac{1}{x}.$$

Identifying $f(x) = 1$ we obtain $u_1' = -x \ln x$ and $u_2' = x$. Then $u_1 = \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x$, $u_2 = \frac{1}{2}x^2$, and

$$y = c_1 + c_2 \ln x + \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x + \frac{1}{2}x^2 \ln x = c_1 + c_2 \ln x + \frac{1}{4}x^2.$$

The initial conditions imply $c_1 + \frac{1}{4} = 1$ and $c_2 + \frac{1}{2} = -\frac{1}{2}$. Thus, $c_1 = \frac{3}{4}$, $c_2 = -1$, and $y = \frac{3}{4} - \ln x + \frac{1}{4}x^2$. The graph is given to the right.



31. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} - 20y = 0.$$

The auxiliary equation is $m^2 + 8m - 20 = (m + 10)(m - 2) = 0$ so that

$$y = c_1 e^{-10t} + c_2 e^{2t} = c_1 x^{-10} + c_2 x^2.$$

33. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 8y = e^{2t}.$$

The auxiliary equation is $m^2 + 9m + 8 = (m + 1)(m + 8) = 0$ so that $y_c = c_1 e^{-t} + c_2 e^{-8t}$. Using undetermined coefficients we try $y_p = Ae^{2t}$. This leads to $30Ae^{2t} = e^{2t}$, so that $A = 1/30$ and

$$y = c_1 e^{-t} + c_2 e^{-8t} + \frac{1}{30}e^{2t} = c_1 x^{-1} + c_2 x^{-8} + \frac{1}{30}x^2.$$

35. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 4 + 3e^t.$$

The auxiliary equation is $m^2 - 4m + 13 = 0$ so that $y_c = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$. Using undetermined coefficients we try $y_p = A + Be^t$. This leads to $13A + 10Be^t = 4 + 3e^t$, so that $A = 4/13$, $B = 3/10$, and

$$\begin{aligned} y &= e^{2t}(c_1 \cos 3t + c_2 \sin 3t) + \frac{4}{13} + \frac{3}{10}e^t \\ &= x^2\left[c_1 \cos(3\ln x) + c_2 \sin(3\ln x)\right] + \frac{4}{13} + \frac{3}{10}x. \end{aligned}$$

we use the substitution $t = -x$ *since the initial conditions are on the interval* $(-\infty, 0)$. *In this case*

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\frac{dy}{dx}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(-\frac{dy}{dx}\right) = -\frac{d}{dt}(y') = -\frac{dy'}{dx}\frac{dx}{dt} = -\frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{d^2y}{dx^2}.$$

37. The differential equation and initial conditions become

$$4t^2\frac{d^2y}{dt^2} + y = 0; \quad y(t)\Big|_{t=1} = 2, \quad y'(t)\Big|_{t=1} = -4.$$

The auxiliary equation is $4m^2 - 4m + 1 = (2m - 1)^2 = 0$, so that

$$y = c_1 t^{1/2} + c_2 t^{1/2}\ln t \quad \text{and} \quad y' = \frac{1}{2}c_1 t^{-1/2} + c_2\left(t^{-1/2} + \frac{1}{2}t^{-1/2}\ln t\right).$$

The initial conditions imply $c_1 = 2$ and $1 + c_2 = -4$. Thus

$$y = 2t^{1/2} - 5t^{1/2}\ln t = 2(-x)^{1/2} - 5(-x)^{1/2}\ln(-x), \quad x < 0.$$

39. Letting $u = x + 2$ we obtain $dy/dx = dy/du$ and, using the Chain Rule,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\right) = \frac{d^2y}{du^2}\frac{du}{dx} = \frac{d^2y}{du^2}(1) = \frac{d^2y}{du^2}.$$

Substituting into the differential equation we obtain

$$u^2\frac{d^2y}{du^2} + u\frac{dy}{du} + y = 0.$$

The auxiliary equation is $m^2 + 1 = 0$ so that

$$y = c_1 \cos(\ln u) + c_2 \sin(\ln u) = c_1 \cos[\ln(x + 2)] + c_2 \sin[\ln(x + 2)].$$

41. For $x^2y'' = 0$ the auxiliary equation is $m(m-1) = 0$ and the general solution is $y = c_1 + c_2x$. The initial conditions imply $c_1 = y_0$ and $c_2 = y_1$, so $y = y_0 + y_1x$. The initial conditions are satisfied for all real values of y_0 and y_1.

For $x^2y'' - 2xy' + 2y = 0$ the auxiliary equation is $m^2 - 3m + 2 = (m-1)(m-2) = 0$ and the general solution is $y = c_1x + c_2x^2$. The initial condition $y(0) = y_0$ implies $0 = y_0$ and the condition $y'(0) = y_1$ implies $c_1 = y_1$. Thus, the initial conditions are satisfied for $y_0 = 0$ and for all real values of y_1.

For $x^2y'' - 4xy' + 6y = 0$ the auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$ and the general solution is $y = c_1x^2 + c_2x^3$. The initial conditions imply $y(0) = 0 = y_0$ and $y'(0) = 0$. Thus, the initial conditions are satisfied only for $y_0 = y_1 = 0$.