1. Irregular singular point: $x = 0$

3. Irregular singular point: $x = 3$; regular singular point: $x = -3$

5. Regular singular points: $x = 0, \pm 2i$

7. Regular singular points: $x = -3, 2$

9. Irregular singular point: $x = 0$; regular singular points: $x = 2, \pm 5$

11. Writing the differential equation in the form

$$y'' + \frac{5}{x-1}y' + \frac{x}{x+1}y = 0$$

we see that $x_0 = 1$ and $x_0 = -1$ are regular singular points. For $x_0 = 1$ the differential equation can be put in the form

$$(x-1)^2 y'' + 5(x-1)y' + \frac{x(x-1)^2}{x+1}y = 0.$$

In this case $p(x) = 5$ and $q(x) = x(x-1)^2/(x+1)$. For $x_0 = -1$ the differential equation can be put in the form

$$(x+1)^2 y'' + 5(x+1)\frac{x+1}{x-1}y' + x(x+1)y = 0.$$

In this case $p(x) = 5(x+1)/(x-1)$ and $q(x) = x(x+1)$.

13. We identify $P(x) = 5/3x + 1$ and $Q(x) = -1/3x^2$, so that $p(x) = xP(x) = \frac{5}{3} + x$ and $q(x) = x^2 Q(x) = -\frac{1}{3}$. Then $a_0 = \frac{5}{3}$, $b_0 = -\frac{1}{3}$, and the indicial equation is

$$r(r-1) + \frac{5}{3}r - \frac{1}{3} = r^2 + \frac{2}{3}r - \frac{1}{3} = \frac{1}{3}(3r^2 + 2r - 1) = \frac{1}{3}(3r-1)(r+1) = 0.$$

The indicial roots are $\frac{1}{3}$ and -1. Since these do not differ by an integer we expect to find two series solutions using the method of Frobenius.

15. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$2xy'' - y' + 2y = \left(2r^2 - 3r\right)c_0 x^{r-1} + \sum_{k=1}^{\infty}[2(k+r-1)(k+r)c_k - (k+r)c_k + 2c_{k-1}]x^{k+r-1} = 0,$$

which implies

$$2r^2 - 3r = r(2r - 3) = 0$$

and

$$(k+r)(2k + 2r - 3)c_k + 2c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 3/2$. For $r = 0$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(2k-3)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = 2c_0, \qquad c_2 = -2c_0, \qquad c_3 = \frac{4}{9}c_0,$$

and so on. For $r = 3/2$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{(2k+3)k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{2}{5}c_0, \qquad c_2 = \frac{2}{35}c_0, \qquad c_3 = -\frac{4}{945}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 + \cdots\right) + C_2 x^{3/2}\left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \cdots\right).$$

17. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$4xy'' + \frac{1}{2}y' + y = \left(4r^2 - \frac{7}{2}r\right)c_0 x^{r-1} + \sum_{k=1}^{\infty}\left[4(k+r)(k+r-1)c_k + \frac{1}{2}(k+r)c_k + c_{k-1}\right]x^{k+r-1}$$

$$= 0,$$

which implies

$$4r^2 - \frac{7}{2}r = r\left(4r - \frac{7}{2}\right) = 0$$

and

$$\frac{1}{2}(k+r)(8k+8r-7)c_k + c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 7/8$. For $r = 0$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(8k-7)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -2c_0, \qquad c_2 = \frac{2}{9}c_0, \qquad c_3 = -\frac{4}{459}c_0,$$

and so on. For $r = 7/8$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{(8k+7)k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{2}{15}c_0, \qquad c_2 = \frac{2}{345}c_0, \qquad c_3 = -\frac{4}{32,085}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right) + C_2 x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32,085}x^3 + \cdots\right).$$

19. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$3xy'' + (2-x)y' - y = \left(3r^2 - r\right) c_0 x^{r-1}$$

$$+ \sum_{k=1}^{\infty} [3(k+r-1)(k+r)c_k + 2(k+r)c_k - (k+r)c_{k-1}]x^{k+r-1}$$

$$= 0,$$

which implies

$$3r^2 - r = r(3r-1) = 0$$

and

$$(k+r)(3k+3r-1)c_k - (k+r)c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 1/3$. For $r = 0$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{3k-1}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{1}{2}c_0, \qquad c_2 = \frac{1}{10}c_0, \qquad c_3 = \frac{1}{80}c_0,$$

and so on. For $r = 1/3$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{3k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{1}{3}c_0, \qquad c_2 = \frac{1}{18}c_0, \qquad c_3 = \frac{1}{162}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 \left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \cdots\right) + C_2 x^{1/3} \left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \cdots\right).$$

21. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$2xy'' - (3+2x)y' + y = \left(2r^2 - 5r\right) c_0 x^{r-1} + \sum_{k=1}^{\infty} [2(k+r)(k+r-1)c_k$$

$$- 3(k+r)c_k - 2(k+r-1)c_{k-1} + c_{k-1}]x^{k+r-1}$$

$$= 0,$$

which implies

$$2r^2 - 5r = r(2r-5) = 0$$

and

$$(k+r)(2k+2r-5)c_k - (2k+2r-3)c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 5/2$. For $r = 0$ the recurrence relation is

$$c_k = \frac{(2k-3)c_{k-1}}{k(2k-5)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{1}{3}c_0, \qquad c_2 = -\frac{1}{6}c_0, \qquad c_3 = -\frac{1}{6}c_0,$$

and so on. For $r = 5/2$ the recurrence relation is

$$c_k = \frac{2(k+1)c_{k-1}}{k(2k+5)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{4}{7}c_0, \qquad c_2 = \frac{4}{21}c_0, \qquad c_3 = \frac{32}{693}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 + \cdots\right) + C_2 x^{5/2}\left(1 + \frac{4}{7}x + \frac{4}{21}x^2 + \frac{32}{693}x^3 + \cdots\right).$$

23. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$9x^2 y'' + 9x^2 y' + 2y = \left(9r^2 - 9r + 2\right) c_0 x^r$$

$$+ \sum_{k=1}^{\infty} [9(k+r)(k+r-1)c_k + 2c_k + 9(k+r-1)c_{k-1}]x^{k+r}$$

$$= 0,$$

which implies

$$9r^2 - 9r + 2 = (3r-1)(3r-2) = 0$$

and

$$[9(k+r)(k+r-1) + 2]c_k + 9(k+r-1)c_{k-1} = 0.$$

The indicial roots are $r = 1/3$ and $r = 2/3$. For $r = 1/3$ the recurrence relation is

$$c_k = -\frac{(3k-2)c_{k-1}}{k(3k-1)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{1}{5}c_0, \qquad c_3 = -\frac{7}{120}c_0,$$

and so on. For $r = 2/3$ the recurrence relation is

$$c_k = -\frac{(3k-1)c_{k-1}}{k(3k+1)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{5}{28}c_0, \qquad c_3 = -\frac{1}{21}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 - \frac{7}{120}x^3 + \cdots\right) + C_2 x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 - \frac{1}{21}x^3 + \cdots\right).$$

25. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + 2y' - xy = \left(r^2 + r\right)c_0 x^{r-1} + \left(r^2 + 3r + 2\right)c_1 x^r$$

$$+ \sum_{k=2}^{\infty}[(k+r)(k+r-1)c_k + 2(k+r)c_k - c_{k-2}]x^{k+r-1}$$

$$= 0,$$

which implies

$$r^2 + r = r(r+1) = 0,$$

$$\left(r^2 + 3r + 2\right)c_1 = 0,$$

and

$$(k+r)(k+r+1)c_k - c_{k-2} = 0.$$

The indicial roots are $r_1 = 0$ and $r_2 = -1$, so $c_1 = 0$. For $r_1 = 0$ the recurrence relation is

$$c_k = \frac{c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = \frac{1}{3!}c_0$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{1}{5!}c_0$$

$$c_{2n} = \frac{1}{(2n+1)!}c_0.$$

For $r_2 = -1$ the recurrence relation is

$$c_k = \frac{c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = \frac{1}{2!}c_0$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{1}{4!}c_0$$

$$c_{2n} = \frac{1}{(2n)!}c_0.$$

The general solution on $(0, \infty)$ is

$$y = C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n} + C_2 x^{-1} \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

$$= \frac{1}{x}\left[C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1} + C_2 \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} \right]$$

$$= \frac{1}{x}[C_1 \sinh x + C_2 \cosh x].$$

27. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' - xy' + y = \left(r^2 - r\right) c_0 x^{r-1} + \sum_{k=0}^{\infty} [(k+r+1)(k+r)c_{k+1} - (k+r)c_k + c_k]x^{k+r} = 0$$

which implies

$$r^2 - r = r(r-1) = 0$$

and

$$(k+r+1)(k+r)c_{k+1} - (k+r-1)c_k = 0.$$

The indicial roots are $r_1 = 1$ and $r_2 = 0$. For $r_1 = 1$ the recurrence relation is

$$c_{k+1} = \frac{kc_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \ldots,$$

and one solution is $y_1 = c_0 x$. A second solution is

$$y_2 = x \int \frac{e^{-\int -1\, dx}}{x^2}\, dx = x \int \frac{e^x}{x^2}\, dx = x \int \frac{1}{x^2} \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots\right) dx$$

$$= x \int \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots\right) dx = x \left[-\frac{1}{x} + \ln x + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{72}x^3 + \cdots\right]$$

$$= x \ln x - 1 + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{72}x^4 + \cdots.$$

The general solution on $(0, \infty)$ is

$$y = C_1 x + C_2 y_2(x).$$

29. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + (1-x)y' - y = r^2 c_0 x^{r-1} + \sum_{k=1}^{\infty} [(k+r)(k+r-1)c_k + (k+r)c_k - (k+r)c_{k-1}]x^{k+r-1} = 0,$$

which implies $r^2 = 0$ and

$$(k+r)^2 c_k - (k+r)c_{k-1} = 0.$$

The indicial roots are $r_1 = r_2 = 0$ and the recurrence relation is

$$c_k = \frac{c_{k-1}}{k}, \quad k = 1, 2, 3, \dots.$$

One solution is

$$y_1 = c_0 \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \right) = c_0 e^x.$$

A second solution is

$$y_2 = y_1 \int \frac{e^{-\int (1/x - 1)dx}}{e^{2x}}\, dx = e^x \int \frac{e^x/x}{e^{2x}}\, dx = e^x \int \frac{1}{x}e^{-x}dx$$

$$= e^x \int \frac{1}{x}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots \right)dx = e^x \int \left(\frac{1}{x} - 1 + \frac{1}{2}x - \frac{1}{3!}x^2 + \cdots \right)dx$$

$$= e^x \left[\ln x - x + \frac{1}{2 \cdot 2}x^2 - \frac{1}{3 \cdot 3!}x^3 + \cdots \right] = e^x \ln x - e^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!}x^n.$$

The general solution on $(0, \infty)$ is

$$y = C_1 e^x + C_2 e^x \left(\ln x - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!}x^n \right).$$

31. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + (x - 6)y' - 3y = (r^2 - 7r)c_0 x^{r-1} + \sum_{k=1}^{\infty}[(k+r)(k+r-1)c_k + (k+r-1)c_{k-1}$$

$$- 6(k+r)c_k - 3c_{k-1}]x^{k+r-1} = 0,$$

which implies

$$r^2 - 7r = r(r-7) = 0$$

and

$$(k+r)(k+r-7)c_k + (k+r-4)c_{k-1} = 0.$$

The indicial roots are $r_1 = 7$ and $r_2 = 0$. For $r_1 = 7$ the recurrence relation is

$$(k+7)kc_k + (k+3)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots ,$$

or

$$c_k = -\frac{k+3}{k(k+7)}c_{k-1}, \quad k = 1, 2, 3, \ldots .$$

Taking $c_0 \neq 0$ we obtain

$$c_1 = -\frac{1}{2}c_0$$

$$c_2 = \frac{5}{18}c_0$$

$$c_3 = -\frac{1}{6}c_0,$$

and so on. Thus, the indicial root $r_1 = 7$ yields a single solution. Now, for $r_2 = 0$ the recurrence relation is

$$k(k-7)c_k + (k-4)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots .$$

Then

$$-6c_1 - 3c_0 = 0$$

$$-10c_2 - 2c_1 = 0$$

$$-12c_3 - c_2 = 0$$

$$-12c_4 + 0c_3 = 0 \implies c_4 = 0$$

$$-10c_5 + c_4 = 0 \implies c_5 = 0$$

$$-6c_6 + 2c_5 = 0 \implies c_6 = 0$$

$$0c_7 + 3c_6 = 0 \implies c_7 \text{ is arbitrary}$$

and

$$c_k = -\frac{k-4}{k(k-7)}c_{k-1}, \quad k = 8, 9, 10, \dots.$$

Taking $c_0 \neq 0$ and $c_7 = 0$ we obtain

$$c_1 = -\frac{1}{2}c_0$$

$$c_2 = \frac{1}{10}c_0$$

$$c_3 = -\frac{1}{120}c_0$$

$$c_4 = c_5 = c_6 = \cdots = 0.$$

Taking $c_0 = 0$ and $c_7 \neq 0$ we obtain

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$$

$$c_8 = -\frac{1}{2}c_7$$

$$c_9 = \frac{5}{36}c_7$$

$$c_{10} = -\frac{1}{36}c_7,$$

and so on. In this case we obtain the two solutions

$$y_1 = 1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3 \quad \text{and} \quad y_2 = x^7 - \frac{1}{2}x^8 + \frac{5}{36}x^9 - \frac{1}{36}x^{10} + \cdots.$$

33. (a) From $t = 1/x$ we have $dt/dx = -1/x^2 = -t^2$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -t^2\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-t^2\frac{dy}{dt}\right) = -t^2\frac{d^2y}{dt^2}\frac{dt}{dx} - \frac{dy}{dt}\left(2t\frac{dt}{dx}\right) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}.$$

Now

$$x^4\frac{d^2y}{dx^2} + \lambda y = \frac{1}{t^4}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \lambda y = \frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \lambda y = 0$$

becomes

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \lambda t y = 0.$$

(b) Substituting $y = \sum_{n=0}^{\infty} c_n t^{n+r}$ into the differential equation and collecting terms, we obtain

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \lambda t y = (r^2 + r)c_0 t^{r-1} + (r^2 + 3r + 2)c_1 t^r$$

$$+ \sum_{k=2}^{\infty}[(k+r)(k+r-1)c_k + 2(k+r)c_k + \lambda c_{k-2}]t^{k+r-1}$$

$$= 0,$$

which implies

$$r^2 + r = r(r+1) = 0,$$

$$\left(r^2 + 3r + 2\right)c_1 = 0,$$

and

$$(k+r)(k+r+1)c_k + \lambda c_{k-2} = 0.$$

The indicial roots are $r_1 = 0$ and $r_2 = -1$, so $c_1 = 0$. For $r_1 = 0$ the recurrence relation is

$$c_k = -\frac{\lambda c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{\lambda}{3!}c_0$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{\lambda^2}{5!}c_0$$

$$\vdots$$

$$c_{2n} = (-1)^n \frac{\lambda^n}{(2n+1)!}c_0.$$

For $r_2 = -1$ the recurrence relation is

$$c_k = -\frac{\lambda c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{\lambda}{2!}c_0$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{\lambda^2}{4!}c_0$$

$$\vdots$$

$$c_{2n} = (-1)^n \frac{\lambda^n}{(2n)!}c_0.$$

The general solution on $(0, \infty)$ is

$$y(t) = c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\sqrt{\lambda}\,t)^{2n} + c_2 t^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\sqrt{\lambda}\,t)^{2n}$$

$$= \frac{1}{t}\left[C_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\sqrt{\lambda}\,t)^{2n+1} + C_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\sqrt{\lambda}\,t)^{2n}\right]$$

$$= \frac{1}{t}[C_1 \sin\sqrt{\lambda}\,t + C_2 \cos\sqrt{\lambda}\,t].$$

(c) Using $t = 1/x$, the solution of the original equation is

$$y(x) = C_1 x \sin\frac{\sqrt{\lambda}}{x} + C_2 x \cos\frac{\sqrt{\lambda}}{x}.$$