1. For $y' - 5y = 0$ an integrating factor is $e^{-\int 5\,dx} = e^{-5x}$ so that $\dfrac{d}{dx}\left[e^{-5x}y\right] = 0$ and $y = ce^{5x}$ for $-\infty < x < \infty$. There is no transient term.

3. For $y' + y = e^{3x}$ an integrating factor is $e^{\int dx} = e^x$ so that $\dfrac{d}{dx}[e^x y] = e^{4x}$ and $y = \frac{1}{4}e^{3x} + ce^{-x}$ for $-\infty < x < \infty$. The transient term is ce^{-x}.

5. For $y' + 3x^2y = x^2$ an integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$ so that $\dfrac{d}{dx}\left[e^{x^3}y\right] = x^2e^{x^3}$ and $y = \frac{1}{3} + ce^{-x^3}$ for $-\infty < x < \infty$. The transient term is ce^{-x^3}.

7. For $y' + \dfrac{1}{x}y = \dfrac{1}{x^2}$ an integrating factor is $e^{\int (1/x)dx} = x$ so that $\dfrac{d}{dx}[xy] = \dfrac{1}{x}$ and $y = \dfrac{1}{x}\ln x + \dfrac{c}{x}$ for $0 < x < \infty$. The entire solution is transient.

9. For $y' - \dfrac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int (1/x)dx} = \dfrac{1}{x}$ so that $\dfrac{d}{dx}\left[\dfrac{1}{x}y\right] = \sin x$ and $y = cx - x\cos x$ for $0 < x < \infty$. There is no transient term.

11. For $y' + \dfrac{4}{x}y = x^2 - 1$ an integrating factor is $e^{\int (4/x)dx} = x^4$ so that $\dfrac{d}{dx}\left[x^4y\right] = x^6 - x^4$ and $y = \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}$ for $0 < x < \infty$. The transient term is cx^{-4}.

13. For $y' + \left(1 + \dfrac{2}{x}\right)y = \dfrac{e^x}{x^2}$ an integrating factor is $e^{\int [1+(2/x)]dx} = x^2e^x$ so that $\dfrac{d}{dx}\left[x^2e^xy\right] = e^{2x}$ and $y = \dfrac{1}{2}\dfrac{e^x}{x^2} + \dfrac{ce^{-x}}{x^2}$ for $0 < x < \infty$. The transient term is $\dfrac{ce^{-x}}{x^2}$.

15. For $\dfrac{dx}{dy} - \dfrac{4}{y}x = 4y^5$ an integrating factor is $e^{-\int (4/y)dy} = e^{\ln y^{-4}} = y^{-4}$ so that $\dfrac{d}{dy}\left[y^{-4}x\right] = 4y$ and $x = 2y^6 + cy^4$ for $0 < y < \infty$. There is no transient term.

17. For $y' + (\tan x)y = \sec x$ an integrating factor is $e^{\int \tan x\,dx} = \sec x$ so that $\dfrac{d}{dx}[(\sec x)\,y] = \sec^2 x$ and $y = \sin x + c\cos x$ for $-\pi/2 < x < \pi/2$. There is no transient term.

19. For $y' + \dfrac{x+2}{x+1}y = \dfrac{2xe^{-x}}{x+1}$ an integrating factor is $e^{\int [(x+2)/(x+1)]dx} = (x+1)e^x$, so $\dfrac{d}{dx}[(x+1)e^xy] = 2x$ and $y = \dfrac{x^2}{x+1}e^{-x} + \dfrac{c}{x+1}e^{-x}$ for $-1 < x < \infty$. The entire solution is transient.

21. For $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$ an integrating factor is $e^{\int \sec\theta\,d\theta} = e^{\ln|\sec x + \tan x|} = \sec\theta + \tan\theta$ so that $\dfrac{d}{d\theta}[(\sec\theta + \tan\theta)r] = 1 + \sin\theta$ and $(\sec\theta + \tan\theta)r = \theta - \cos\theta + c$ for $-\pi/2 < \theta < \pi/2$.

23. For $y' + \left(3 + \frac{1}{x}\right) y = \frac{e^{-3x}}{x}$ an integrating factor is $e^{\int[3+(1/x)]dx} = xe^{3x}$ so that $\frac{d}{dx}\left[xe^{3x}y\right] = 1$ and $y = e^{-3x} + \frac{ce^{-3x}}{x}$ for $0 < x < \infty$. The entire solution is transient.

25. For $y' + \frac{1}{x} y = \frac{1}{x} e^x$ an integrating factor is $e^{\int(1/x)dx} = x$ so that $\frac{d}{dx}[xy] = e^x$ and $y = \frac{1}{x}e^x + \frac{c}{x}$ for $0 < x < \infty$. If $y(1) = 2$ then $c = 2 - e$ and $y = \frac{1}{x}e^x + \frac{2-e}{x}$.

27. For $\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$ an integrating factor is $e^{\int(R/L)\,dt} = e^{Rt/L}$ so that $\frac{d}{dt}\left[e^{Rt/L} i\right] = \frac{E}{L}e^{Rt/L}$ and $i = \frac{E}{R} + ce^{-Rt/L}$ for $-\infty < t < \infty$. If $i(0) = i_0$ then $c = i_0 - E/R$ and $i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{-Rt/L}$.

29. For $y' + \frac{1}{x+1} y = \frac{\ln x}{x+1}$ an integrating factor is $e^{\int[1/(x+1)]dx} = x + 1$ so that $\frac{d}{dx}[(x+1)y] = \ln x$ and $y = \frac{x}{x+1} \ln x - \frac{x}{x+1} + \frac{c}{x+1}$ for $0 < x < \infty$. If $y(1) = 10$ then $c = 21$ and

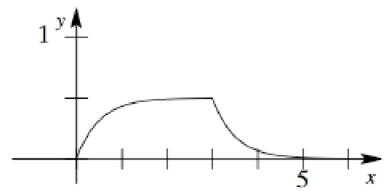
$$y = \frac{x}{x+1} \ln x - \frac{x}{x+1} + \frac{21}{x+1}.$$

31. For $y' + 2y = f(x)$ an integrating factor is e^{2x} so that

$$ye^{2x} = \begin{cases} \frac{1}{2}e^{2x} + c_1, & 0 \le x \le 3 \\ c_2, & x > 3. \end{cases}$$

If $y(0) = 0$ then $c_1 = -1/2$ and for continuity we must have $c_2 = \frac{1}{2}e^6 - \frac{1}{2}$ so that

$$y = \begin{cases} \frac{1}{2}(1 - e^{-2x}), & 0 \le x \le 3 \\ \frac{1}{2}(e^6 - 1)e^{-2x}, & x > 3. \end{cases}$$

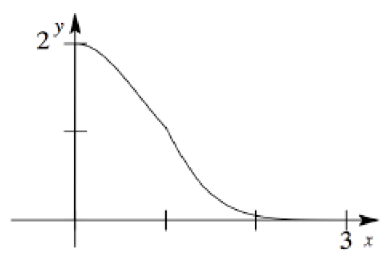


33. For $y' + 2xy = f(x)$ an integrating factor is e^{x^2} so that

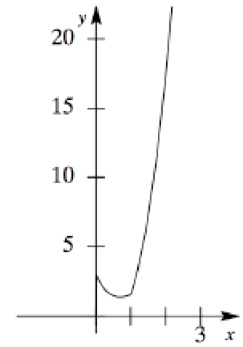
$$ye^{x^2} = \begin{cases} \frac{1}{2}e^{x^2} + c_1, & 0 \le x \le 1 \\ c_2, & x > 1. \end{cases}$$

If $y(0) = 2$ then $c_1 = 3/2$ and for continuity we must have $c_2 = \frac{1}{2}e + \frac{3}{2}$ so that

$$y = \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-x^2}, & 0 \le x \le 1 \\ \left(\frac{1}{2}e + \frac{3}{2}\right) e^{-x^2}, & x > 1. \end{cases}$$



35. We first solve the initial-value problem $y' + 2y = 4x$, $y(0) = 3$ on the interval $[0, 1]$. The integrating factor is $e^{\int 2\,dx} = e^{2x}$, so



$$\frac{d}{dx}[e^{2x}y] = 4xe^{2x}$$
$$e^{2x}y = \int 4xe^{2x}dx = 2xe^{2x} - e^{2x} + c_1$$
$$y = 2x - 1 + c_1e^{-2x}.$$

Using the initial condition, we find $y(0) = -1 + c_1 = 3$, so $c_1 = 4$ and $y = 2x - 1 + 4e^{-2x}$, $0 \le x \le 1$. Now, since $y(1) = 2 - 1 + 4e^{-2} = 1 + 4e^{-2}$, we solve the initial-value problem $y' - (2/x)y = 4x$, $y(1) = 1 + 4e^{-2}$ on the interval $(1, \infty)$. The integrating factor is $e^{\int(-2/x)dx} = e^{-2\ln x} = x^{-2}$, so

$$\frac{d}{dx}[x^{-2}y] = 4xx^{-2} = \frac{4}{x}$$
$$x^{-2}y = \int \frac{4}{x}\,dx = 4\ln x + c_2$$
$$y = 4x^2\ln x + c_2x^2.$$

(We use $\ln x$ instead of $\ln|x|$ because $x > 1$.) Using the initial condition we find $y(1) = c_2 = 1 + 4e^{-2}$, so $y = 4x^2\ln x + (1 + 4e^{-2})x^2$, $x > 1$. Thus, the solution of the original initial-value problem is

$$y = \begin{cases} 2x - 1 + 4e^{-2x}, & 0 \le x \le 1 \\ 4x^2\ln x + (1 + 4e^{-2})x^2, & x > 1. \end{cases}$$

See Problem 42 in this section.

47. Writing the differential equation as $\dfrac{dE}{dt} + \dfrac{1}{RC}E = 0$ we see that an integrating factor is $e^{t/RC}$. Then

$$\frac{d}{dt}[e^{t/RC}E] = 0$$
$$e^{t/RC}E = c$$
$$E = ce^{-t/RC}.$$

From $E(4) = ce^{-4/RC} = E_0$ we find $c = E_0e^{4/RC}$. Thus, the solution of the initial-value problem is

$$E = E_0e^{4/RC}e^{-t/RC} = E_0e^{-(t-4)/RC}.$$