r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

   (a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

   (c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

   (d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

2. A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

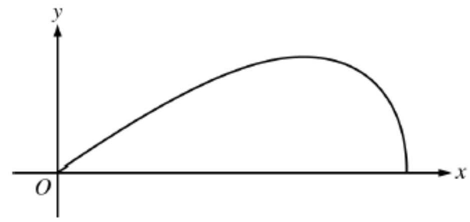
   A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

   (a) Find the positions of particles P and Q at time $t = 1$.

   (b) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.

   (c) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.

   (d) Find the total distance traveled by particle P over the time interval $0 \le t \le \pi$.

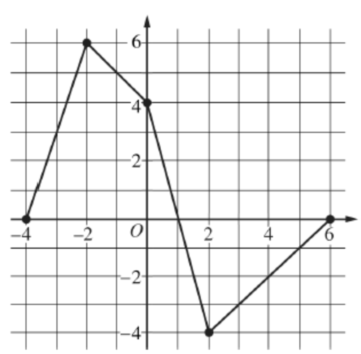


3. A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

   (a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

   (b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

   (c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?



Graph of f

4. Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\, dt$.

   (a) On what open intervals is the graph of G concave up? Give a reason for your answer.

   (b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

   (c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

   (d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$.

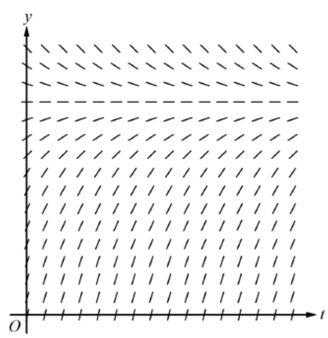
(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

6. A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(a) A portion of the slope field for the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.



(b) Using correct units, interpret the statement $\lim_{t \to \infty} A(t) = 12$ in the context of this problem.

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ with initial condition $A(0) = 0$.

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\dfrac{dy}{dt} = 3 - \dfrac{y}{t + 2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

2021 AB Free Response Answers

1)
   a) 8 mg/cm^2/cm; At distance of 2.25 cm, $f'(2.25)$ is rate of change of density.
   b) $269\pi = 845.088$
   c) Overestimate because $rf(r)$ is increasing
   d) 2.497

2)
   a) $x_P(1) = 5.371$, $x_Q(1) = 8.564$
   b) Toward; P is to the left and moving right ($v_P(1) > 0$), Q is to the right and moving left ($v_Q(1) < 0$)
   c) $a_Q(1) = 1.027$; speed decreasing because $v_Q(1)$ and $a_Q(1)$ have opposite signs
   d) 1.931

3)
   a) 16
   b) 0.6
   c) $\sqrt{\frac{15}{32}}$

4)
   a) $-4 < x < -2$ and $2 < x < 6$; f is increasing
   b) 5.5
   c) -2
   d) $\frac{8}{3}$, yes because G is differentiable and continuous on the interval

5)
   a)
   b) $y = \frac{1}{4}x + \sqrt{3}$
   c) $\left(\frac{\pi}{2}, 2\right)$
   d) Relative max., $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

6)
   a) →→→
   b) Over time, amount of medication approaches 12 mg.
   c) $y = 12 - 12e^{-t/3}$
   d) Decreasing, $\frac{d^2y}{dx^2} < 0$ at $t = 1$