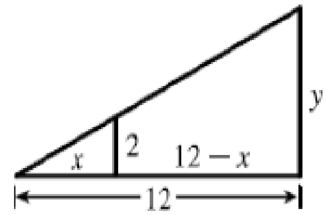
<u>Practice Related Rates</u>

p. 280: 20-21, 23-29, 31-32, 34-35, 38

20. We are given that $\frac{dx}{dt}=1.6$ m/s. By similar triangles, $\frac{y}{12}=\frac{2}{x}\Rightarrow \quad y=\frac{24}{x}\Rightarrow$

$\frac{dy}{dt}=-\frac{24}{x^2}\frac{dx}{dt}=-\frac{24}{x^2}(1.6)$. When $x=8$, $\frac{dy}{dt}=-\frac{24(1.6)}{64}=-0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.



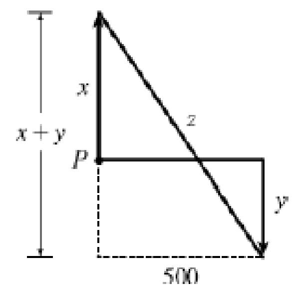
21. We are given that $\frac{dx}{dt}=4$ ft/s and $\frac{dy}{dt}=5$ ft/s. $z^2=(x+y)^2+500^2\Rightarrow$

$2z\frac{dz}{dt}=2(x+y)\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$. Fifteen minutes after the woman starts, we have

$x=(4\text{ ft/s})(60\text{ s/min})=4800$ ft and $y=5\cdot 15\cdot 60=4500\Rightarrow$

$z=\sqrt{(4800+4500)^2+500^2}=\sqrt{86,740,000}$, so

$z\frac{dz}{dt}=(x+y)\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=\frac{4800+4500}{\sqrt{86,740,000}}(4+5)=\frac{837}{\sqrt{8674}}\approx 8.987$ ft/s.



23. We are given that $V=\frac{4}{3}\pi r^3$.

(a) $V=288\pi\text{ in}^3=\frac{4}{3}\pi r^3\Rightarrow 288\left(\frac{3}{4}\right)=r^3\ \Rightarrow 216=r^3\ \Rightarrow r=6$ in.

(b) We are given that $\frac{dr}{dt}=0.15$ in/s, and $r=6$.

We need to find $\frac{dV}{dt}=\frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}=4\pi(6)^2(0.15)=21.6\pi\text{ in}^3/\text{s}$.

(c) Now we are given that $\frac{dV}{dt}=-3.6\text{ in}^3/\text{s}$, with $V=288\pi\text{ in}^3$ and $r=6$.

Now $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=\frac{dV}{dt}\frac{1}{4\pi r^2}\Rightarrow\ \frac{dr}{dt}=\frac{-3.6}{4\pi\cdot 6^2}=\frac{-0.025}{\pi}$ in/s ≈ -0.00797 in/s.

24. Let t be the time in hours, x be the distance traveled by Train A, y be the distance traveled by Train B, and z be the distance between the trains. We are given that $\frac{dx}{dt}=30$ mph, and $\frac{dy}{dt}=40$ mph. We also know that at noon (time 0), $x=0$, and $y=z=10$ m.

(a) At 12:30 PM, $t=\frac{1}{2}$, so $x=0+\frac{1}{2}(40)=20$, and $y=10+\frac{1}{2}(30)=25$. Therefore the distance between the trains at this time is $z=\sqrt{x^2+y^2}=\sqrt{20^2+25^2}=\sqrt{1025}=5\sqrt{41}$ miles.

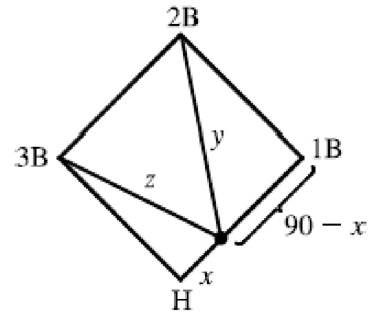
(b) $z^2=x^2+y^2\Rightarrow 2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}\Rightarrow\frac{dz}{dt}=\frac{1}{z}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$. At 12:30 PM,

$\frac{dz}{dt}=\frac{1}{5\sqrt{41}}(20(40)+25(30))=\frac{1550}{5\sqrt{41}}=\frac{310}{\sqrt{41}}$ mph.

(c) Train A is $40a$ miles from the intersection after a hours and train B is $10+30a$ miles from the intersection. Their distances are equal when $40a=10+30a\Rightarrow a=1$ hour, or 1 PM. At this time,

$\frac{dz}{dt}=\frac{1}{z}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)=\frac{1}{40\sqrt{2}}(40\cdot 40+30\cdot 40)=\frac{70}{\sqrt{2}}\approx 49.498$ mph.

25. We are given that $\frac{dx}{dt} = 24$ ft/s.



(a) $y^2 = (90-x)^2 + 90^2 \Rightarrow\ 2y\frac{dy}{dt} = 2(90-x)\left(-\frac{dx}{dt}\right)$. When $x = 45$,

$y = \sqrt{45^2+90^2} = 45\sqrt{5}$, so $\frac{dy}{dt} = \frac{90-x}{y}\left(-\frac{dx}{dt}\right) = \frac{45}{45\sqrt{5}}(-24) = \frac{-24}{\sqrt{5}}$; the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.733$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer – and we do.

$z^2 = x^2 + 90^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt}$. When $x = 45, z = 45\sqrt{5}$, so $\frac{dz}{dt} = \frac{45}{45\sqrt{5}}(24) = \frac{24}{\sqrt{5}} \approx 10.733$ ft/s.

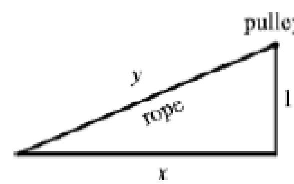
26. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\frac{dh}{dt} = 1$ cm/min and

$\frac{dA}{dt} = 2\ \text{cm}^2/\text{min}$. Using the Product Rule, we have $\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$. When $h = 10$ and $A = 100$,

we have $100 = \frac{1}{2}b(10) \Rightarrow \frac{1}{2}b = 10 \Rightarrow\ b = 20$, so $2 = \frac{1}{2}\left(20\cdot 1 + 10\frac{db}{dt}\right) \Rightarrow\ 4 = 20 + 10\frac{db}{dt} \Rightarrow$

$\frac{db}{dt} = \frac{4-20}{10} = -1.6$ cm/min.

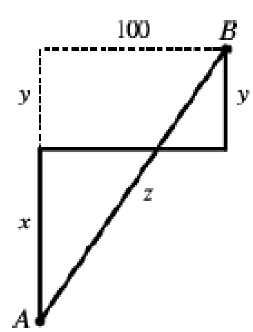
27. Given $\frac{dy}{dt} = -1$ m/s, find $\frac{dx}{dt}$ when $x = 8$ m.



$y^2 = x^2 + 1 \Rightarrow\ 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow\ \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$.

When $x = 8,\ y = \sqrt{65}$, so $\frac{dx}{dt} = -\frac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8} \approx 1.008$ m/s.

28. We are given that $\frac{dx}{dt} = 35$ km/h and $\frac{dy}{dt} = 325$ km/h.



$z^2 = (x+y)^2 + 100^2 \Rightarrow 2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$. At 4:00 PM,

$x = 4(350 = 140$ and $y = 4(25) = 100 \Rightarrow$

$z = \sqrt{(140+100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so $\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$

$= \frac{140+100}{260}(35+25)\ \frac{720}{13} \approx 55.385$ km/h.

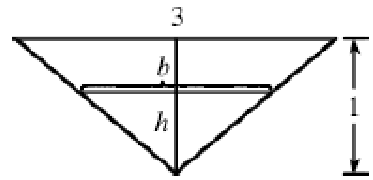
29. The distance z of the particle to the origin is given by $z=\sqrt{x^2+y^2}$, so $z^2=x^2+[2\sin(\pi x/2)]^2 \Rightarrow$

$$2z\frac{dz}{dt}=2x\frac{dx}{dt}+4\cdot 2\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}\frac{dx}{dt} \Rightarrow \quad z\frac{dz}{dt}=x\frac{dx}{dt}+2\pi\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right)\frac{dx}{dt}.$$

When $(x,y)=\left(\frac{1}{3},1\right)$, $z=\sqrt{\left(\frac{1}{3}\right)^2+1^2}=\sqrt{\frac{10}{9}}=\frac{1}{3}\sqrt{10}$, so $\frac{1}{3}\sqrt{10}\frac{dz}{dt}=\frac{1}{3}\sqrt{10}+2\pi\sin\frac{\pi}{6}\cos\frac{\pi}{6}\cdot\sqrt{10} \Rightarrow$

$$\frac{1}{3}\frac{dz}{dt}=\frac{1}{3}+2\pi\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{3}\right) \Rightarrow \frac{dz}{dt}=1+\frac{3\sqrt{3}\pi}{2}\text{ cm/s.}$$

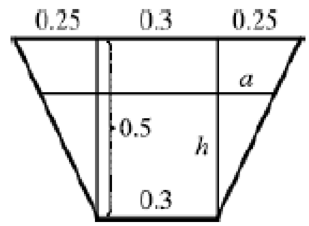
31. By similar triangles, $\frac{3}{1}=\frac{b}{h}$, so $b=3h$. The trough has volume



$$V=\tfrac{1}{2}bh(10)=5(3h)h=15h^2 \Rightarrow \quad 12=\frac{dV}{dt}=30h\frac{dh}{dt} \Rightarrow \quad \frac{dh}{dt}=\frac{2}{5h}.$$

When $h=\frac{1}{2}$, $\frac{dh}{dt}=\frac{2}{5\cdot\frac{1}{2}}=\frac{4}{5}$ ft/min.

32. The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1+\text{base}_2)\cdot(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V=(10)\frac{1}{2}[0.3+(0.3+2a)]h$.

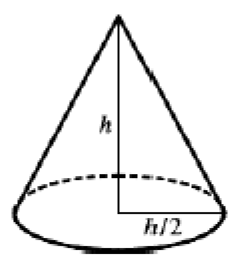


By similar triangles, $\frac{a}{h}=\frac{0.25}{0.5}=\frac{1}{2}$, so $2a=h \Rightarrow$

$V=5(0.6+h)h=3h+5h^2$. Now $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt} \Rightarrow 0.2=(3+10h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{0.2}{3+10h}$. When

$h=0.3$, $\frac{dh}{dt}=\frac{0.2}{3+10(0.3)}=\frac{0.2}{6}$ m/min $=\frac{1}{30}$ m/min or $=\frac{10}{3}$ cm/min.

34. We are given that $\frac{dV}{dt}=30\text{ ft}^3/\text{min}$. $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2h=\frac{\pi h^3}{2} \Rightarrow$

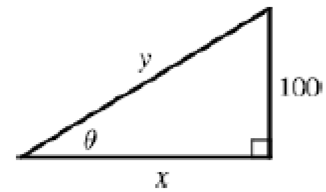


$$\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt} \Rightarrow \quad 30=\frac{\pi h^2}{4}\frac{dh}{dt} \Rightarrow \quad \frac{dh}{dt}=\frac{120}{\pi h^2}.$$ When $h=10$ ft,

$$\frac{dh}{dt}=\frac{120}{10^2\pi}=\frac{6}{5\pi}\approx 0.382\text{ ft/min.}$$

35. We are given that $\frac{dx}{dt} = 8$ ft/s.

Then $\cot\theta = \frac{x}{100} \Rightarrow x = 100\cot\theta \Rightarrow \frac{dx}{dt} = -100\csc^2\theta\frac{d\theta}{dt} \Rightarrow$



$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{100}\cdot 8$. When $y = 200$, $\sin\theta = \frac{100}{200} = \frac{1}{2} \Rightarrow$

$\frac{d\theta}{dt} = -\frac{(1/2)^2}{100}\cdot 8 = -\frac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

38. From the figure and given information, we have $x^2 + y^2 = L^2$,

$\frac{dy}{dt} = -0.15$ m/s, and $\frac{dx}{dt} = 0.2$ m/s when $x = 3$ m. Differentiating implicitly

with respect to t, we get $x^2 + y^2 = L^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow y\frac{dy}{dt} = -x\frac{dx}{dt}$.

Substituting the given information gives us $y(-0.15) = -3(0.2) \Rightarrow y = 4$ m.

Thus, $3^2 + 4^2 = L^2 \Rightarrow L^2 = 25 \Rightarrow L = 5$ m.