Second FTOC

p. 442: 11-23 odd, 49-53, 55-56, 71-75 odd, 76-81, 86

11. $f(t)=\sqrt{t+t^3}$ and $g(x)=\int_0^x \sqrt{t+t^3}\,dt$, so by the FTC1,

$g'(x)=f(x)=\sqrt{x+x^3}$.

13. $f(t)=(t-t^2)^8$ and $g(s)=\int_5^s (t-t^2)^8 dt$, so by the FTC1, $g'(s)=f(s)=(s-s^2)^8$.

15. $F(x)=\int_x^0 \sqrt{1+\sec t}\,dt=-\int_0^x \sqrt{1+\sec t}\,dt \Rightarrow F'(x)=-\dfrac{d}{dx}\int_0^x \sqrt{1+\sec t}\,dt=-\sqrt{1+\sec x}$

17. Let $u=e^x$. Then $\dfrac{du}{dx}=e^x$. Also $\dfrac{dh}{dx}=\dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x)=\frac{d}{dx}\int_1^{e^x}\ln t\,dt=\frac{d}{du}\int_1^u \ln t\,dt\cdot\frac{du}{dx}=\ln u\frac{du}{dx}=(\ln e^x)\cdot e^x=xe^x$$

19. Let $u=3x+2$. Then $\dfrac{du}{dx}=3$. Also $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y'=\frac{d}{dx}\int_1^{3x+2}\frac{t}{1+t^3}dt=\frac{d}{du}\int_1^u\frac{t}{1+t^3}dt\cdot\frac{du}{dx}=\frac{u}{1+u^3}\frac{du}{dx}=\frac{3x+2}{1+(3x+2)^3}\cdot 3=\frac{3(3x+2)}{1+(3x+2)^3}$$

21. $u=\sqrt{x}$. Then $\dfrac{du}{dx}=\dfrac{1}{2\sqrt{x}}$. Also $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y'=\frac{d}{dx}\int_{\sqrt{x}}^{\pi/4}\theta\tan\theta\,d\theta=-\frac{d}{du}\int_{\pi/4}^{\sqrt{x}}\theta\tan\theta\,d\theta\cdot\frac{du}{dx}=-u\tan u\frac{du}{dx}=-\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}}=-\frac{1}{2}\tan\sqrt{x}$$

23. $$\frac{d}{dx}\left[\int_{-x^2}^{x^2}\frac{1}{1+t^2}dt\right]=\frac{d}{dx}\left[\int_{-x^2}^{0}\frac{1}{1+t^2}dt\right]+\frac{d}{dx}\left[\int_{0}^{x^2}\frac{1}{1+t^2}dt\right]=-\frac{d}{dx}\left[\int_{0}^{-x^2}\frac{1}{1+t^2}dt\right]+\frac{d}{dx}\left[\int_{0}^{x^2}\frac{1}{1+t^2}dt\right]$$

$$=-\frac{-2x}{1+(-x^2)^2}+\frac{2x}{1+(x^2)^2}=\frac{2x}{1+x^4}+\frac{2x}{1+x^4}=\frac{4x}{1+x^4}$$, option (**B**).

49. For $k>0$, $\int_k^{k^2}\frac{1}{x}dx=\ln x\Big]_k^{k^2}=\ln k^2-\ln k=\ln k$, option (**D**).

50. $$g(0)+g(1)+g(2)+g(3)=\int_0^0 f(t)dt+\int_0^1 f(t)dt+\int_0^2 f(t)dt+\int_0^3 f(t)dt$$
$$=0+\left(1\cdot1+\tfrac{1}{2}\cdot1\cdot\right)+\left[g(1)+1\cdot2\right]+\left[g(2)+\tfrac{1}{2}\cdot1\cdot2\right]$$
$$=0+1.5+\left[1.5+2\right]+\left[g(2)+1\right]$$
$$=0+1.5+3.5+\left[3.5+1\right]=0+1.5+3.5+4.5=9.5$$, option (**D**).

51. (a) If $G(x)=\int_0^{x^2} f(t)\,dt$, then $G'(x)=f\left(x^2\right)\cdot 2x \Rightarrow G'(2)=f\left(2^2\right)\cdot 2(2)=4\cdot f(4)=4\cdot1=4$

(b) G has a maximum where $G'(x)=0$ and G' changes from positive to negative. $G'(x)=0 \Leftrightarrow$ $2x\cdot f\left(x^2\right)=0 \Rightarrow x=0$ or $f\left(x^2\right)=0$. From the graph, $f(x)=0 \Leftrightarrow x=3 \Rightarrow f\left(x^2\right)=0 \Leftrightarrow x=\sqrt{3}$. When $x<\sqrt{3}$, $G'(x)=2x\cdot f\left(x^2\right)<0$ and when $x>\sqrt{3}$, $G'(x)=2x\cdot f\left(x^2\right)>0$, so G has a minimum at $x=\sqrt{3}$.

52. (a) If $h(x)=\int_0^{x^2} f(t)\,dt$, then $h'(x)=f\left(x^2\right)\cdot\left(x^2\right)'=f\left(x^2\right)\cdot 2x$.

(b) h has a minimum where $h'(x)=0$ and h' changes from positive to negative. $h'(x)=0 \Leftrightarrow$ $2x\cdot f\left(x^2\right)=0 \Rightarrow x=0$ or $f\left(x^2\right)=0$. From the graph, $f(x)=0 \Rightarrow x=5 \Rightarrow f\left(x^2\right)=0 \Leftrightarrow x=\sqrt{5}$, or $f(x)=0 \Rightarrow x=10 \Rightarrow f\left(x^2\right)=0 \Leftrightarrow x=\sqrt{10}$. When $x<\sqrt{5}$, $h'(x)=2x\cdot f\left(x^2\right)<0$ and when $x>\sqrt{5}$, $h'(x)=2x\cdot f\left(x^2\right)>0$, so G has a minimum at $x=\sqrt{5}$. Since $h'(x)>0$ for $\sqrt{5}<x<\sqrt{10}$, h cannot have a minimum at $x=\sqrt{10}$.

53. (a) $A=\int_0^k x\cdot\sqrt{25-x^2}\,dx \Rightarrow \frac{dA}{dk}=\frac{d}{dk}\left[\int_0^k x\cdot\sqrt{25-x^2}\,dx\right]=k\sqrt{25-k^2}$. $\frac{dA}{dk}$ is maximized at either endpoints or critical points. We first find a critical point: $\frac{d^2A}{dx^2}=k\frac{-2k}{2\sqrt{25-k^2}}+\sqrt{25-k^2}(1)=$

$\left.\frac{dA}{dk}\right|_{k=0}=0\sqrt{25-0^2}=0$, $\left.\frac{dA}{dk}\right|_{k=5}=5\sqrt{25-(5)^2}=0$, and $\left.\frac{dA}{dk}\right|_{k=\sqrt{25/2}}>0$

So max at $k=\sqrt{\frac{25}{2}}$.

(b) $A=\int_0^k x\cdot\sqrt{25-x^2}\,dx \Rightarrow \frac{dA}{dt}=\frac{d}{dt}\left[\int_0^k x\cdot\sqrt{25-x^2}\,dx\right]=k\cdot\sqrt{25-k^2}\cdot\frac{dk}{dt}=k\cdot\sqrt{25-k^2}\cdot 2$.

When $k=3$, $\frac{dA}{dx}=2\cdot 3\cdot\sqrt{25-3^2}=6\sqrt{16}=6\cdot 4=24$ units^2/sec.

55. $R=\int_0^k \frac{\sin\left(\frac{\pi}{2}x\right)}{x}dx \Rightarrow \frac{dR}{dt}=\frac{d}{dt}\left[\int_0^k \frac{\sin\left(\frac{\pi}{2}x\right)}{x}dx\right]=\frac{\sin\left(\frac{\pi}{2}k\right)}{k}\cdot\frac{dk}{dt}=\frac{\sin\left(\frac{\pi}{2}k\right)}{k}\cdot(0.5)$.

When $k=1.5$, $\frac{dR}{dt}=0.5\frac{\sin\left(\frac{\pi}{2}\cdot\frac{3}{2}\right)}{1.5}=\frac{\sin\left(\frac{3\pi}{4}\right)}{2(1.5)}=\frac{\sqrt{2}}{6}$ units^2/min, option (C).

56. $F(x)=\int_1^x f(t)\,dt=0 \Leftrightarrow f(x)-f(1)=0 \Leftrightarrow f(x)=f(1)=\left|2(1)-10\right|-4=4$.

$f(x)=\left|2x-10\right|-4=4 \Leftrightarrow \left|2x-10\right|=8 \Leftrightarrow 2x-10=8$, or $2x-10=-8$.

$2x-10=8 \Rightarrow 2x=18 \Rightarrow x=9$, and $2x-10=-8 \Rightarrow 2x=2 \Rightarrow x=1$. In addition, looking at the area under the curve $\int_1^3 f(t)\,dt=-\int_3^5 f(t)\,dt \Leftrightarrow \int_1^5 f(t)\,dt=0$, Thus $F(m)=0$ when $m=1,5$, or 9.

71. $g(x)=\int_{2x}^{3x}\frac{u^2-1}{u^2+1}du=\int_{2x}^{0}\frac{u^2-1}{u^2+1}du+\int_0^{3x}\frac{u^2-1}{u^2+1}du=-\int_0^{2x}\frac{u^2-1}{u^2+1}du+\int_0^{3x}\frac{u^2-1}{u^2+1}\Rightarrow$

$g'(x)=\frac{(2x)^2-1}{(2x)^2+1}\cdot\frac{d}{dx}(2x)+\frac{(3x)^2-1}{(3x)^2+1}\cdot\frac{d}{dx}(3x)=-2\cdot\frac{4x^2-1}{4x^2+1}+3\cdot\frac{9x^2-1}{9x^2+1}$

73. $F(x)=\int_x^{x^2}e^{t^2}\,dt=\int_x^0 e^{t^2}\,dt+\int_0^{x^2}e^{t^2}\,dt=-=\int_0^x e^{t^2}\,dt+\int_0^{x^2}e^{t^2}\,dt \Rightarrow$

$F'(x)=-e^{x^2}+e^{(x^2)^2}\cdot\frac{d}{dx}\left(x^2\right)=-e^{x^2}+2xe^{x^4}$

75. $y=\int_{\cos x}^{\sin x}\ln(1+2v)\,dv=\int_{\cos x}^{0}\ln(1+2v)\,dvt+\int_{0}^{\sin x}\ln(1+2v)\,dv$

$$=-\int_{0}^{\cos x}\ln(1+2v)\,dv+\int_{0}^{\sin x}\ln(1+2v)\,dv\Rightarrow$$

76. $f(x)=\int_0^x(1-t)^2dt$ is increasing when $f'(x)=(1-x^2)$ is positive. Then $f'(x)>0\Leftrightarrow 1-x^2>0$ $\Leftrightarrow |x|<1$, so f is increasing on $(-1,1)$.

77. $y=\int_0^x\frac{t^2}{t^2+t+2}dt\Rightarrow y'=\frac{x^2}{x^2+x+2}\Rightarrow$

$$y''=\frac{(x^2+x+2)(2x)-x^2(2x+1)}{(x^2+x+2)^2}=\frac{2x^3+2x^2+4x-2x^3-x^2}{(x^2+x+2)^2}=\frac{x^2+4x}{(x^2+x+2)^2}=\frac{x(x+4)}{(x^2+x+2)^2}.$$

The curve y is concave down when $y''<0$; that is, on the interval $(-4,0)$.

78. If $F(x)=-4x+\int_2^x\sqrt{7+r^2}\,dr$ then

(A) $F(2)=-4(2)+0\neq 0$, so this statement is *not* true.

(B) $F'(x)=-4+\frac{d}{dx}\left[\int_2^x\sqrt{7+r^2}\,dr\right]=-4+\sqrt{7+x^2}$, and $F'(3)=-4+\sqrt{7+3^2}=-4+4=0$, so this statement is *true*.

(C) $F'(x)=-4+\frac{d}{dx}\left[\int_2^x\sqrt{7+r^2}\,dr\right]=-4+\sqrt{7+x^2}\Rightarrow F'(x)>0$ when $-4+\sqrt{7+x^2}>0\Rightarrow$ $\sqrt{7+x^2}>4\Rightarrow 7+x^2>16\Rightarrow x^2>9\Leftrightarrow x>3$ or $x<-3$. Therefore, statement (C) is *not true*.

(D) $F'(x)=-4+\sqrt{7+x^2}\Rightarrow F''(x)=\frac{2x}{2\sqrt{7+x^2}}=\frac{x}{\sqrt{7+x^2}}$. $F''(0)=0$ and when $x<0$, $F''(x)<0$, and when $x>0$, $F''(x)>0$, so F has an inflection point at $x=0$. Statement (D) is *not true*. Only statement **(B)** is true.

79. (a) The average value of g over the interval $-2\le x\le 4$ is

$$\frac{1}{4-(-2)}\int_{-2}^{4}g(x)\,dx=\frac{1}{6}\left[\int_{-2}^{0}g(x)\,dx+\int_{0}^{8}g(x)\,dx+\int_{8}^{10}g(x)\,dx\right]=\frac{1}{6}\left[\left(\tfrac12\cdot2\cdot5\right)+\left(-\tfrac12\pi\cdot4^2\right)+\left(\tfrac12\cdot2\cdot4\right)\right]$$

$$=\tfrac16\left[5+(-8\pi)+(4)\right]=\frac{9-8\pi}{6}\approx-2.6888.$$

(b) The average rate of change of g over the interval $-2\le x\le 4$ is $\frac{g(4)-g(-2)}{4-(-2)}=\frac{-4-5}{6}=-\frac{9}{6}=-\frac{3}{2}$.

(c) If $J'(x)=g(x)$, then $J(x)=\int_{-3}^{x}g(t)\,dt\Rightarrow J(8)=J(-3)+\int_{-3}^{0}g(t)\,dt+\int_{0}^{8}g(t)\,dt$

$$=5+\left(\tfrac12\cdot3\cdot5\right)+\left(-\tfrac12\pi4^2\right)=5+\tfrac{15}{2}-8\pi=\tfrac{25}{2}-8\pi\approx-12.633.$$

(d) F has a minimum where $F'(x)=0$ and F' changes from negative to positive. If $F(x)=\int_0^x g(t)\,dt$ then $F'(x)=\frac{d}{dx}\left[\int_0^x g(t)\,dt\right]=g(x)$. Using the graph of g, we see that $F'(x)=g(x)=0$ when $x=-3$, $x=0$, and $x=8$. The graph of g also shows that $F'(x)=g(x)$ changes from negative to positive only at $x=8$, so the absolute minimum value of F occurs when $x=8$ and the minimum value is $F(8)=\int_0^8 g(t)\,dt=-\tfrac12\pi\cdot4^2=-8\pi$.

(e) Inflection points occur when F changes concavity, which occurs when $F''(x)$ changes sign. $F(x)=\int_0^x g(t)\,dt \Rightarrow F'(x)=g(x) \Rightarrow F''(x)=g'(x)$. Based on the graph of g, $F''(x)=g'(x)<0$ for $-2<x<0$ and $0<x<4$, and $F''(x)=g'(x)>0$ for $-4<x<-2$, $4<x<8$, and $8<x<10$. Thus F changes concavity at $x=-2$ and $x=4$; hence, F has inflection points at $x=-2$ and $x=4$.

(f) $H(x)=\int_0^{x^2} g(t)\,dt \Rightarrow H'(x)=g\left(x^2\right)\cdot 2x \Rightarrow H'(3)=g\left(3^2\right)\cdot 2(3)=6\cdot g(9)=6\cdot 2=12.$

$H'(x)=2x\cdot g(x^2) \Rightarrow H''(x)=2x\cdot g'(x^2)\cdot 2x+g(x^2)\cdot 2=4x^2g'(x^2)+2g(x^2) \Rightarrow$

$H''(3)=4\cdot(3)^2g'(3^2)+2g(3^2)=36\cdot g'(9)+2g(9)=36\cdot 2+2\cdot 2=76$

80. If $F(x)=\int_1^x f(t)\,dt$, then by FTC1, $F'(x)=f(x)$, and also, $F''(x)=f'(x)$. F is concave down where F'' is negative; that is, where f' is negative. The given graph shows that f is decreasing $\left(f'<0\right)$ on the interval $(-1,1)$.

81. $F(x)=\int_2^x e^{t^2}\,dt \Rightarrow F'(x)=e^{x^2}$, so the slope at $x=2$ is $e^{2^2}=e^4$. The y-coordinate of the point on F at $x=2$ is $F(2)=\int_2^2 e^{t^2}\,dt=0$ since the limits are equal. An equation of the tangent line is $y=e^4(x-2)+0$ or $y=e^4x-2e^4$.

$g'(y)=f(y)$, and $g''(y)=f'(y)$; $g'(y)=\int_0^{\sin y}\sqrt{1+t^2}\,dt \Leftrightarrow g''(y)=\sqrt{1+\sin^2 y}(\cos y)$. Thus $g''\left(\frac{\pi}{6}\right)=\sqrt{1+\frac{1}{4}}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{15}}{2}$

By FTC2, $\int_1^4 f'(x)\,dx=f(4)-f(1)$, so $17=f(4)-12 \Rightarrow f(4)=17+12=29$.

86. (a) By FTC1, $g'(x)=f(x)$. So, $g'(x)=f(x)=0$ at $x=2,4,6,8,$ and 10. g has local maxima at $x=2$ and $x=6$ (since $f=g'$ changes from positive to negative there) and local minima at $x=4$ and $x=8$. There is no local maximum or minimum at $x=10$ since f is not defined for $x>10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right|>\left|\int_2^4 f\,dt\right|>\left|\int_4^6 f\,dt\right|>\left|\int_6^8 f\,dt\right|>\left|\int_8^{10} f\,dt\right|$.

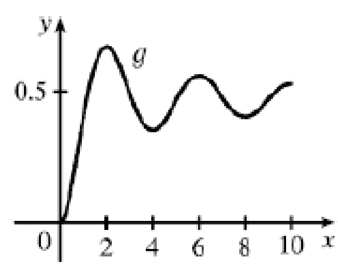
So $g(2)=\left|\int_0^2 f\,dt\right|$, $g(6)=\int_0^6 f\,dt=g(2)-\left|\int_2^4 f\,dt\right|+\left|\int_4^6 f\,dt\right|$,

and $g(10)=\int_0^{10} f\,dt=g(6)-\left|\int_6^8 f\,dt\right|+\left|\int_8^{10} f\,dt\right|$.

Thus, $g(2)>g(6)>g(10)$, and so the absolute maximum of $g(x)$ occurs at $x=2$.

(c) g is concave down on those intervals where $g''<0$. But $g'(x)=f(x)$, so $g''(x)=f'(x)$, which is negative on (approximately) $(1,3),(5,7)$, and $(9,10)$. So g is concave down on these intervals.

(d)



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9. $\int_0^1 f(x)\,dx - \int_1^2 f(x)\,dx = [F(1)-F(0)]-[F(2)-F(1)] = (-1-2)-(3-(-1)) = -3-4 = -7$, (**C**).

10. $\int_1^x f(t)\,dt = F(x)-F(1) = F(x)-(-2) = F(x)+2$, which is choice (**C**).

11. The amount of water that is added to the tank in the first 12 minutes is $\int_0^{12} R(t)\,dt = \int_0^{12} 120e^{-0.2t}\,dt = \frac{120}{-0.2}e^{-0.2t}\Big]_0^{12} = -600e^{-0.2t}\Big]_0^{12} = -600\left(e^{-2.4}-e\right) \approx 545.57$ gallons. So the total amount of water in the tank 12 minutes later is $500+546=1046$ gallons, option (**A**).

12. The number of people left 9 minutes after the conclusion of the game is

$$2000-\int_0^9 D(t)\,dt = 2000-\int_0^9 \frac{800}{(t+1)^2}\,dt = 2000-800\int_1^{10}\frac{1}{u^2}\,du \qquad \begin{bmatrix} u=t+1 \\ du=dt \end{bmatrix}$$

$$= 2000-800\left[-\frac{1}{u}\right]_1^{10} = 2000+800\left(\frac{1}{10}-\left(\frac{1}{1}\right)\right) = 2000-720 = 1280, \text{ option } (\mathbf{D}).$$

13. f is increasing when $f'(x) = \frac{d}{dx}\int_1^{2x}\sin\left(t^2\right)dt > 0 \Leftrightarrow \sin(2x)^2\cdot 2 > 0 \Leftrightarrow \sin\left(4x^2\right) > 0$. But on $[0,1.5]$, $\sin\left(4x^2\right) > 0$ for $0 < x < 0.886$ and $1.253 < x \le 1.5$, so f is increasing on $(0,0.886)\cup(1.253,1.5]$.

14. $F(x) = \int_1^x t^2\cos\left(t^2\right)dt \Rightarrow F'(x) = x^2\cos\left(x^2\right) \Rightarrow$

$F''(x) = x^2\left[-\sin\left(x^2\right)\cdot 2x\right]+\cos\left(x^2\right)\cdot 2x = 2x\left[\cos\left(x^2\right)-x^2\sin\left(x^2\right)\right]$.

For $-2<x<2$, $F''(x)=0 \Rightarrow x=0$, $x=a_1\approx -1.851, a_2=-0.928, a_3=0.928$, and $x=a_4\approx 1.851$. $F''(x)$ changes sign at each of these values, so F has 5 inflection points in this open interval.