New seats today…
You may sit where you wish

At 8:15, we’re going to exchange books at the library

Definitions and Terms
A differential equation (diff. eq., DE) is an equation that involves \( x, y, \) and some derivatives of \( y \):

\[
\frac{dy}{dx} + 5y = e^x \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0
\]

\( x + y'' = 2xy' + y \)

These are called ordinary differential equations (ODEs) because \( y \) is a function of only \( x \).

The order of a DE is the highest derivative in the equation.

\[
\frac{d^3y}{dx^3} + \left( \frac{dy}{dx} \right)^3 = 0 \quad \rightarrow \text{Order 3}
\]

Equations using partial derivatives are called partial differential equations (PDEs).

\[
\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = 0 \quad \frac{\partial u}{\partial y} + \sin x \frac{\partial u}{\partial x} = 0
\]

We will only be studying ODEs in this course.

A DE is linear if it is linear in \( y, y', y'', \ldots, y^{(n)} \)
i.e. Each term has a coefficient that is a function of only \( x \).

<table>
<thead>
<tr>
<th>Linear</th>
<th>Non-linear</th>
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<tbody>
<tr>
<td>( y'' - 2y' + y = 0 )</td>
<td>( (y - 1)y' + 2y = e^x )</td>
</tr>
<tr>
<td>( \frac{d^3y}{dx^3} + x^2 \frac{dy}{dx} - 5y = e^x )</td>
<td>( \frac{d^2y}{dx^2} + \sin y = 0 )</td>
</tr>
<tr>
<td>( (y - x)dx + 4xdy = 0 )</td>
<td>( \frac{d^4y}{dx^4} + y^2 = 0 )</td>
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</tbody>
</table>
y(x), defined on an interval I, is a solution of a DE if, when substituted into the DE, it reduces the equation to an identity.

Ex. Verify that the given function is a solution to the given DE.
\[ y' = xy^{\frac{1}{2}}, \ y = \frac{1}{16}x^4 \]

Ex. Verify that the given function is a solution to the given DE.
\[ y'' - 2y' + y = 0, \ y = xe^x \]

Notice that \( y = 0 \) is a solution to both DEs. This is called the trivial solution.

Ex. Find values for \( m \) that would make \( y = e^{mx} \) a solution of the DE \( 2y'' + 7y' - 4y = 0 \).

The interval on which a solution is defined is called the interval of definition.

a.k.a. interval of existence
a.k.a. interval of validity
a.k.a. domain of the solution

Ex. Given \( x^2 + y^2 = 25 \), find \( \frac{dy}{dx} \)

Consider \( y = \frac{1}{x} \), which is a solution to the DE \( xy' + y = 0 \).

As a function, \( y = \frac{1}{x} \) has domain \( (-\infty, 0) \cup (0, \infty) \).

A solution must be defined on an interval, so we must choose the interval of definition, either \( (-\infty, 0) \) or \( (0, \infty) \) or some subinterval. Which one we choose depends on other info that we could be given.
So \(x^2 + y^2 = 25\) is a solution to the differential equation \(\frac{dy}{dx} = \frac{x}{y}\). This is called an implicit solution.

The explicit solution could be \(y = \sqrt{25 - x^2}\) or \(y = -\sqrt{25 - x^2}\).

The one we choose depends on other information that we may be given.

Not all implicit solutions can be written explicitly.

Note that \(x^2 + y^2 = c\) would also be a solution for the DE \(\frac{dy}{dx} = \frac{x}{y}\) for any \(c \geq 0\).

\(x^2 + y^2 = c\) is called a one-parameter family of solutions.

When solving an \(n\)th order DE, we will want to find an \(n\)-parameter family of solutions.

Ex. Show that \(x = c_1 \sin 4t\) is a solution to the linear DE \(x'' + 16x = 0\).

A solution where we’ve chosen a value for the parameter is called a particular solution.

With the parameters, we call it the general solution.

\(x^2 + y^2 = 25\) was a particular solution

\(x^2 + y^2 = c\) was a general solution

Ex. Show that \(y = c_1 e^x + c_2 xe^x\) is a family of solutions of the DE \(y'' - 2y' + y = 0\).

Ex. We saw that \(y = \frac{1}{16} x^4\) was a solution to \(y' = xy^{1/2}\). A family of solutions is \(y = (\frac{1}{4} x^2 + c)^2\).

Note that we get our particular solution by setting \(c = 0\).

But we saw that \(y = 0\) is also a solution, and it’s not a member of the family of solutions.

This extra solution is called a singular solution.
Note that \( y = cx^4 \) is a family of solutions of the DE \( xy' - 4y = 0 \).

However, a singular solution could be

\[
y = \begin{cases} -x^4 & x > 0 \\ x^4 & x \leq 0 \end{cases}
\]

When we find a family of solutions, how can we know if it describes all solutions or if there are singular solutions…

Ex. Verify that \( x = e^{-5t}, y = 2e^{-5t} \) is a solution to the system of DEs

\[
\begin{align*}
\frac{dx}{dt} &= 3x - 4y \\
\frac{dy}{dt} &= 4x - 7y
\end{align*}
\]

is called a system of DEs. The solution is \( x = f(t), y = g(t) \).