Linear Independence

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is linearly dependent if there exist constants $x_1, x_2, \ldots, x_p$ (not all zero) such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

$A\vec{x} = \vec{0}$

→ This equation is called a linear dependence relation.

→ The set is linearly independent if $x_1 = x_2 = \ldots = x_p = 0$ is the only solution.

Ex. Determine if the vectors are dependent. Find a linear dependence relation. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ ← soln. that's non-trivial

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \to -3R_1 + R_3]{R_2 \to -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{-6}R_3]{R_2 \to \frac{1}{-3}R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_2 - R_3]{} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↓ free variable ∴ more than just triv. soln.

$$\xrightarrow{R_1 \to -4R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{array}{l} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \end{array} \Longrightarrow \begin{array}{l} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{array}$$

$$\Longrightarrow \vec{x} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} \xrightarrow{\text{pick } x_3} \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

dependence relation
$2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

∴ vectors are dependent

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

→ Note this is the same as our homogeneous equation $A\mathbf{x} = \mathbf{0}$, where the vectors are the columns of A.

Thm. The following are equivalent:

i. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

ii. The columns of A are linearly independent

iii. The linear system with augmented matrix $[A \mid \mathbf{0}]$ has no free variables

iv. A has a pivot in each column

Ex. Determine if the vectors are dependent.

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

Thm. A set of two or more vectors is linearly dependent if and only if at least one is a linear combination of the others.

Ex. Assume a vector **w** is in Span{**u**,**v**}. → set of all lin. combinations of $\vec{u}$ and $\vec{v}$

Describe **w** and show that **u**, **v**, and **w** are linearly dependent.

$\vec{w} = a\vec{u} + b\vec{v}$

$\_\_\ \vec{u} + \_\_\ \vec{v} + \_\_\ \vec{w} = \vec{0}$

$\underline{-a}\,\vec{u} + \underline{-b}\,\vec{v} + \underline{1}\,(a\vec{u} + b\vec{v}) = \vec{0}$

$\therefore$ vectors are dep.

Ex. Determine if the vectors are dependent. ↓

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \xrightarrow[R_2 \to -3R_2 + R_1]{} \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}$

no pivot in a col. $\therefore$ dep.

$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{0}$

$\vec{v}_2 = \frac{-x_1}{x_2} \vec{v}_1$

If dep., $\vec{v}_2 = c\vec{v}_1$.

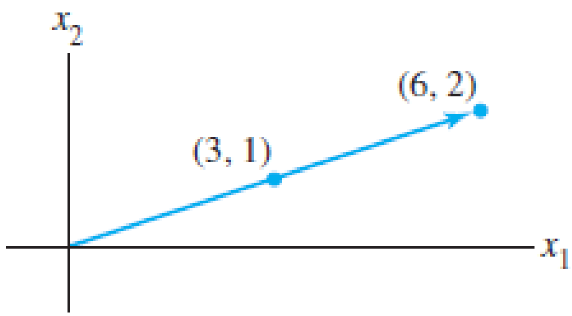
$\vec{v}_2 = 2\vec{v}_1$

b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ 2 & 2 \end{bmatrix} \xrightarrow[R_2 \to -2R_1 + 3R_2]{} \begin{bmatrix} 3 & 6 \\ 0 & -6 \end{bmatrix}$

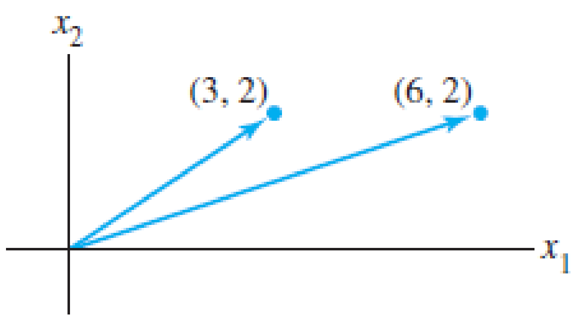
pivot in every col. $\therefore$ indep.

$\cancel{\vec{v}_2 \stackrel{?}{=} \vec{v}_1}$

Two vectors are linearly dependent if one is a multiple of the other.



Linearly dependent



Linearly independent

Note: This doesn't work for more than 2 vectors!

Thm. If a set contains more vectors than there are entries in each vector, then the set is dependent.

Ex. Show that the set is dependent. $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

pivot in every col.

no

Thm. If a set contains the zero vector, then the set is dependent.

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{0}\}$$

$$\underline{0}\,\vec{v}_1 + \underline{0}\,\vec{v}_2 + \underline{0}\,\vec{v}_3 + \underline{0}\,\vec{v}_4 + \underline{5}\,\vec{0} = \vec{0}$$

Ex. Determine if the set is dependent.

a. $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \right\}$ too many

b. $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\}$

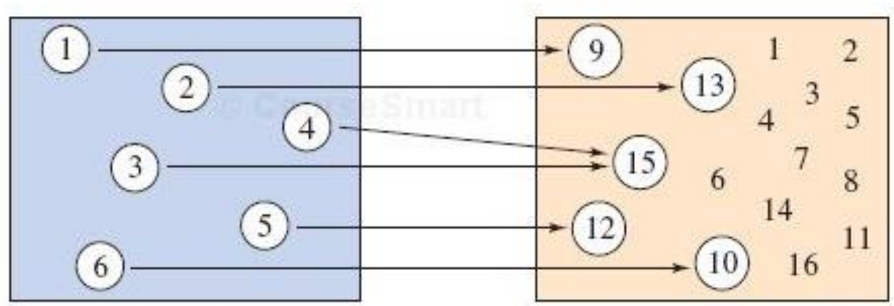
$\vec{0}$

c. $\left\{ \underbrace{\begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}}_{\vec{v}_2} \right\}$

$\vec{v}_2 = -3\vec{v}_1$

Intro to Linear Transformations

Def. A function f from set A to set B is a relation that assigns to each element x in set A exactly one element y in set B.



Set A Set B

Domain Codomain

$$\underset{A}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}\underset{\mathbb{R}^4}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{\mathbb{R}^2}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}$$

$$A\mathbf{x} = \mathbf{b}$$

domain

We can think of A as transforming $\mathbf{x}$ in $\mathbb{R}^4$ to $\mathbf{b}$ in $\mathbb{R}^2$.

codomain

A <u>transformation</u> (or <u>function</u> or <u>mapping</u>) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

$T: \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = 7x$

$\mathbb{R}^n$ is the domain

$\mathbb{R}^m$ is the codomain

The set of all $T(\mathbf{x})$ is called the <u>range</u>

→ The range is a subset of the codomain

The rest of this section will focus on mappings associated with matrix multiplication

$T(\vec{x}) = A\vec{x}$

$$\mathbf{x} \mapsto A\mathbf{x}$$

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

a. If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $T(\mathbf{u})$.

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ -7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

$f(x) = 7x$

$f(3) =$

$3 \longrightarrow ?$

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by
$T(\mathbf{x}) = A\mathbf{x}$.

$A\vec{x} = \vec{b}$

b. If $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, find an $\mathbf{x}$ whose image under T is $\mathbf{b}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \xrightarrow[R_3 \to R_1 + R_3]{R_2 \to -3R_1 + R_2} \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{4}R_3]{R_2 \to \frac{1}{14}R_2} \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 1 & -1/2 \end{array}\right] \xrightarrow[R_3 \to R_2 - R_3]{} \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$f(x) = 7x$

$7x = 28$

$? \longrightarrow 28$

$$\xrightarrow{R_1 \to 3R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{matrix} x_1 = 3/2 \\ x_2 = -1/2 \end{matrix} \Longrightarrow \vec{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

Was this answer unique?

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

c. If $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, find an $\mathbf{x}$ whose image under T is $\mathbf{c}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & \underline{\underline{-5/2}} \end{array}\right]$$

no soln.

$$? \longrightarrow \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

Ex. Define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

d. Find all $\mathbf{x}$ that are mapped into the zero vector.

$$A\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 3 & 5 & 0 \\ -1 & 7 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 14 & 0 \\ 0 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

Ex. Find the image of $\mathbf{x}$ under the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix} \to \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}}$$

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$$

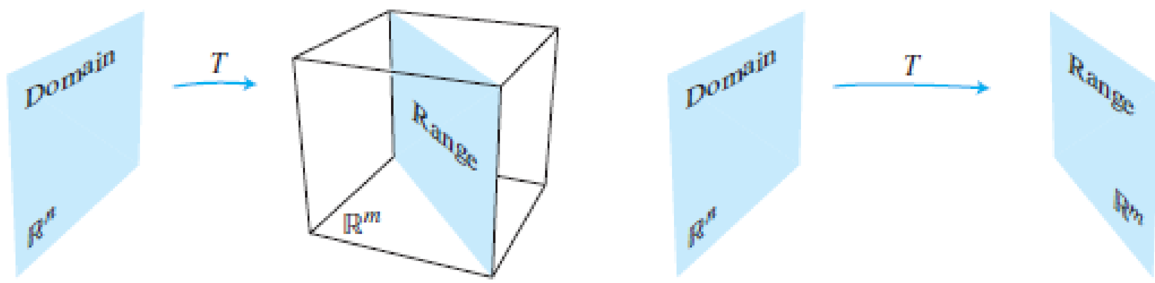
This projects the point onto the x_1x_2-plane.

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every **b** in $\mathbb{R}^m$ is the image of *at least* one **x** in $\mathbb{R}^n$.

$A\vec{x} = \vec{b}$ has soln. for every $\vec{b}$.

→ The range makes up the entire codomain

→ Every vector in $\mathbb{R}^m$ is the output at least once



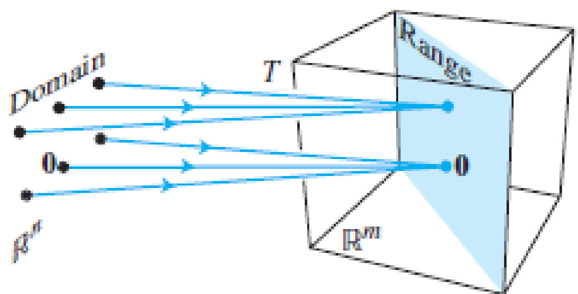
T is *not* onto $\mathbb{R}^m$ $\qquad$ T is onto $\mathbb{R}^m$

A mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is <u>one-to-one</u> if every **b** in $\mathbb{R}^m$ is the image of *at most* one **x** in $\mathbb{R}^n$.

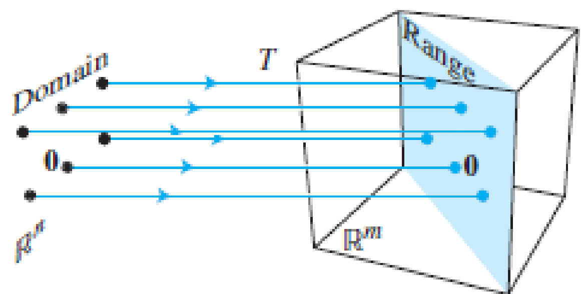
→ Every vector in the range is an output exactly once

→ Not all vectors in $\mathbb{R}^m$ are outputs

→ $T(\mathbf{x})$ has either a unique solution or no solution



T is *not* one-to-one



T *is* one-to-one

Ex. Define $T: \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T one-to-one?

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} ?$$

onto?

$A\vec{x} = \vec{b}$ has solution for any $\vec{b}$?

→ yes, pivot in every row

one-to-one?

If $A\vec{x} = \vec{b}$ has solution, it is unique?

→ no, not pivot in every column

We remember properties of vector/matrix/scalar addition and multiplication:

Distributive: $A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Commutative: $A(c\mathbf{u}) = cA(\mathbf{u})$

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

These lead to the properties of a linear transformation T.

$T: f \rightarrow f$

$\frac{d}{dx}(f+g) = f' + g'$

$\frac{d}{dx}(cf) = cf'$

For any linear transformation,

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

In particular, $T(\mathbf{0}) = \mathbf{0}$.

→ This can be generalized to be true for any number of vectors. This is called the superposition principle.

Ex. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = 3\mathbf{x}$. Show that T is a linear transformation.

$$T(\vec{u}+\vec{v}) = 3(\vec{u}+\vec{v}) = 3\vec{u} + 3\vec{v} = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = 3(c\vec{u}) = c(3\vec{u}) = cT(\vec{u})$$

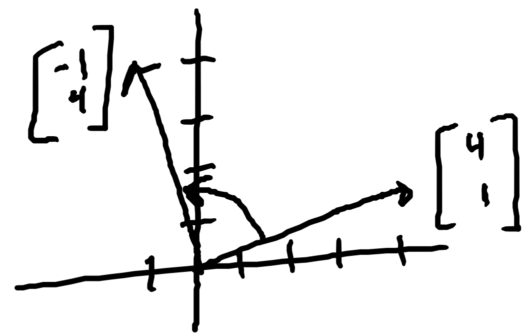
What does this transformation represent graphically?

Ex. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$.

Find $T(\mathbf{u})$:

a) $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\to$ $T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

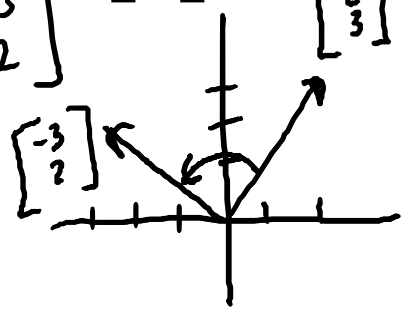


b) $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\to$ $T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$= 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \to \begin{bmatrix} -3 \\ 2 \end{bmatrix}$



What does this transformation represent graphically?

Matrix of a Linear Transformation

We have been talking about different linear transformations, not just ones that are matrix multiplication.

In fact, all linear transformations can be represented by a matrix multiplication.

To find the matrix, we will be using the columns of I_n, which we will call $\mathbf{e}_1$, $\mathbf{e}_2$, etc.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underbrace{\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \underset{\hat{i}}{\mathbf{e}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underset{\hat{j}}{\mathbf{e}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underset{\hat{k}}{\mathbf{e}_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These are called the standard basis vectors of $\mathbb{R}^3$.

<u>Ex.</u> Suppose T is a linear transformation such that $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. Describe the image of an arbitrary $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right) = T\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(x_1\vec{e}_1 + x_2\vec{e}_2\right)$$

$$= T\left(x_1\vec{e}_1\right) + T\left(x_2\vec{e}_2\right) = x_1 T\left(\vec{e}_1\right) + x_2 T\left(\vec{e}_2\right)$$

$$= x_1\begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thm. If $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, there is a unique $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$.

→ The columns of A will be the transformation of the columns of I. In other words:

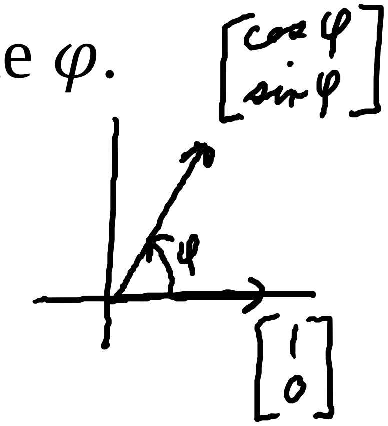
$$A = [T(\mathbf{e}_1) \quad \dots \quad T(\mathbf{e}_n)]$$

→ This is called the standard matrix for the linear transformation.

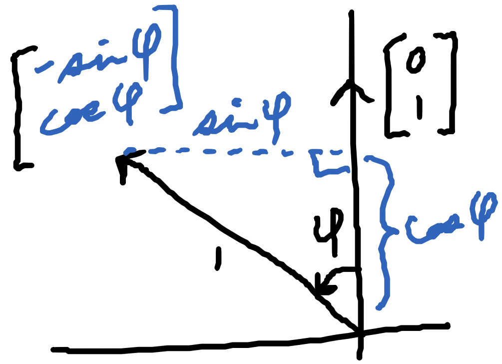
→ Please note mapping $\mathbb{R}^n \to \mathbb{R}^m$ requires a matrix that is $m \times n$.

Ex. Find the standard matrix for the transformation that rotates each point in $\mathbb{R}^2$ counterclockwise about the origin through an angle φ.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}$$



$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix}$$



$$\boxed{A=\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}}$$

p. 73-75 has the standard matrices for several common geometric linear transformations.

→ Even more transformations come from the composition of transformations.

Ex. Define $T: \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T one-to-one?

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

onto?

yes, pivot in every row

one-to-one

no, not pivot in every column

Thm. Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. The following are equivalent:

i. T is one-to-one.

ii. A has a pivot in each column.

iii. A has no free variables.

iv. The columns of A are linearly independent.

v. The equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

→ This links us with all of the equivalent statements from last class.

Thm. Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. The following are equivalent:

i. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

ii. A has a pivot in each row.

iii. The columns of A span $\mathbb{R}^m$.

iv. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ in $\mathbb{R}^m$.

v. Every $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A

Ex. Let $T(x_1,x_2) = (3x_1 + x_2,\ 5x_1 + 7x_2,\ x_1 + 3x_2)$.
Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? Is T one-to-one?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 3 & 1 \end{bmatrix} \xrightarrow[R_3 \to -3R_1 + R_3]{R_2 \to -5R_1 + R_2} \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & -8 \end{bmatrix} \xrightarrow[R_3 \to R_2 - R_3]{} \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} ? \\ \\ \end{bmatrix}$$

onto? → no, not pivot in every row

one-to-one? → yes, pivot in every col.