Math 205 – Calculus III

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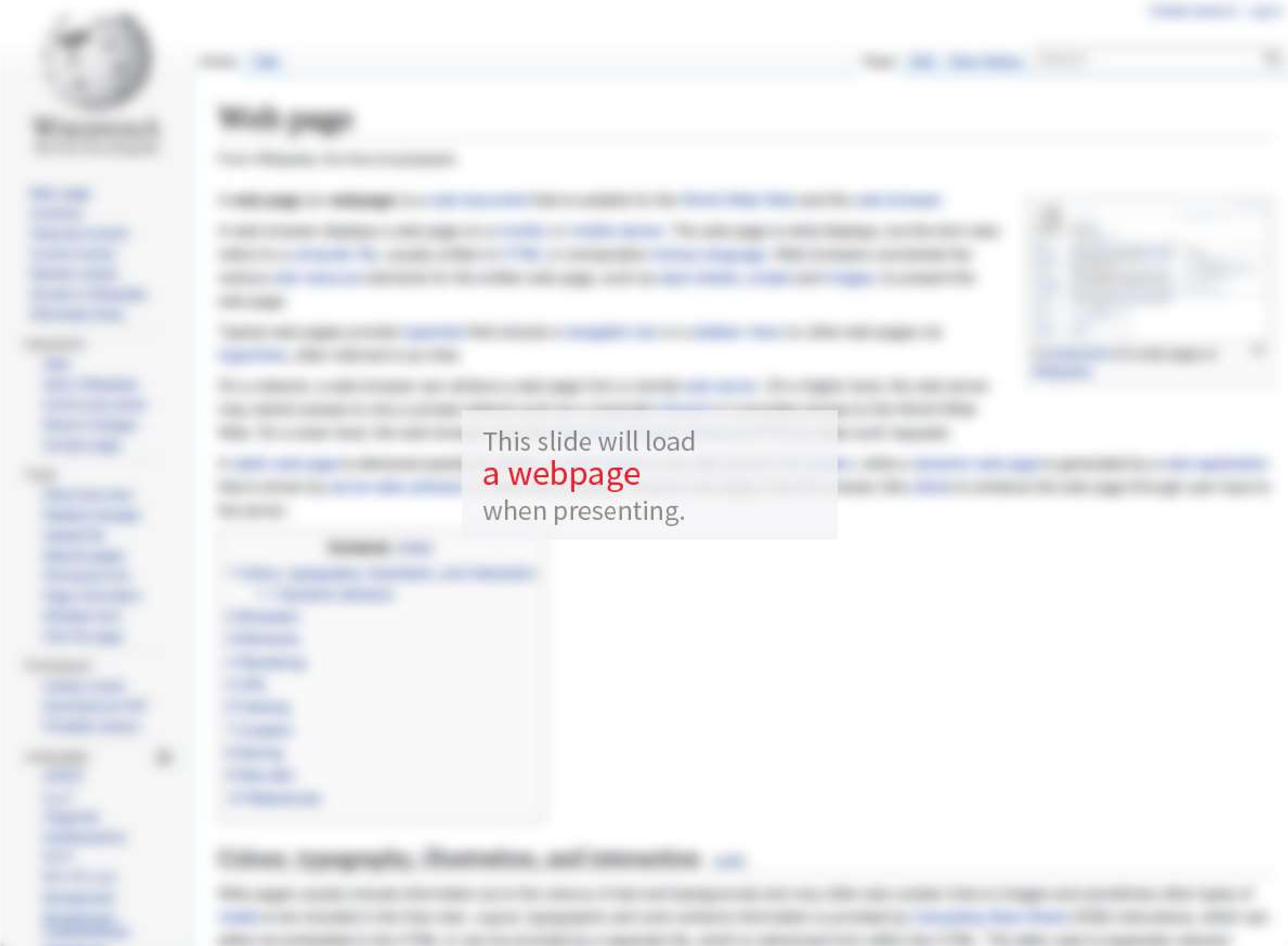
[If you are trying to crash the course, that will be the first thing we talk about.]

Student Learning Outcomes

Students will be able to

- perform calculus on vector valued functions.
- perform vector operations using geometry in space.
- perform calculus on multivariable functions.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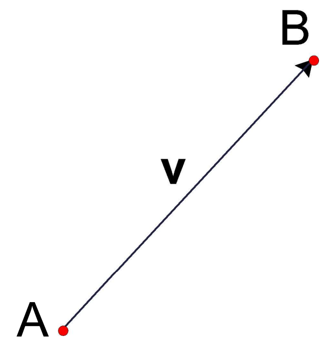
Vectors

<u>Def.</u> A <u>vector</u> is a quantity that has both magnitude and direction.

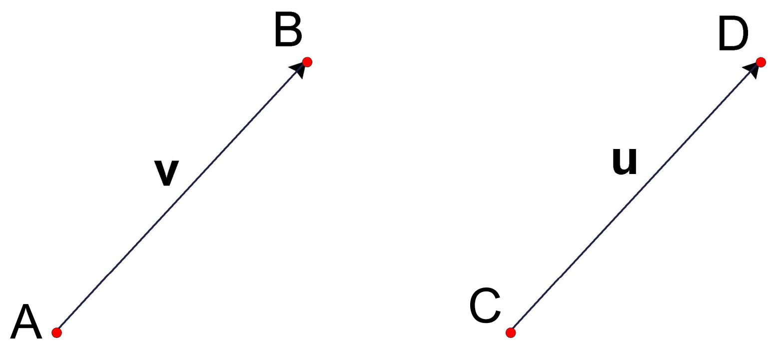


$$\overrightarrow{AB} \text{ or } \vec{v} \text{ or } \mathbf{v}$$

v is <u>displacement vector</u> from *A* to *B*

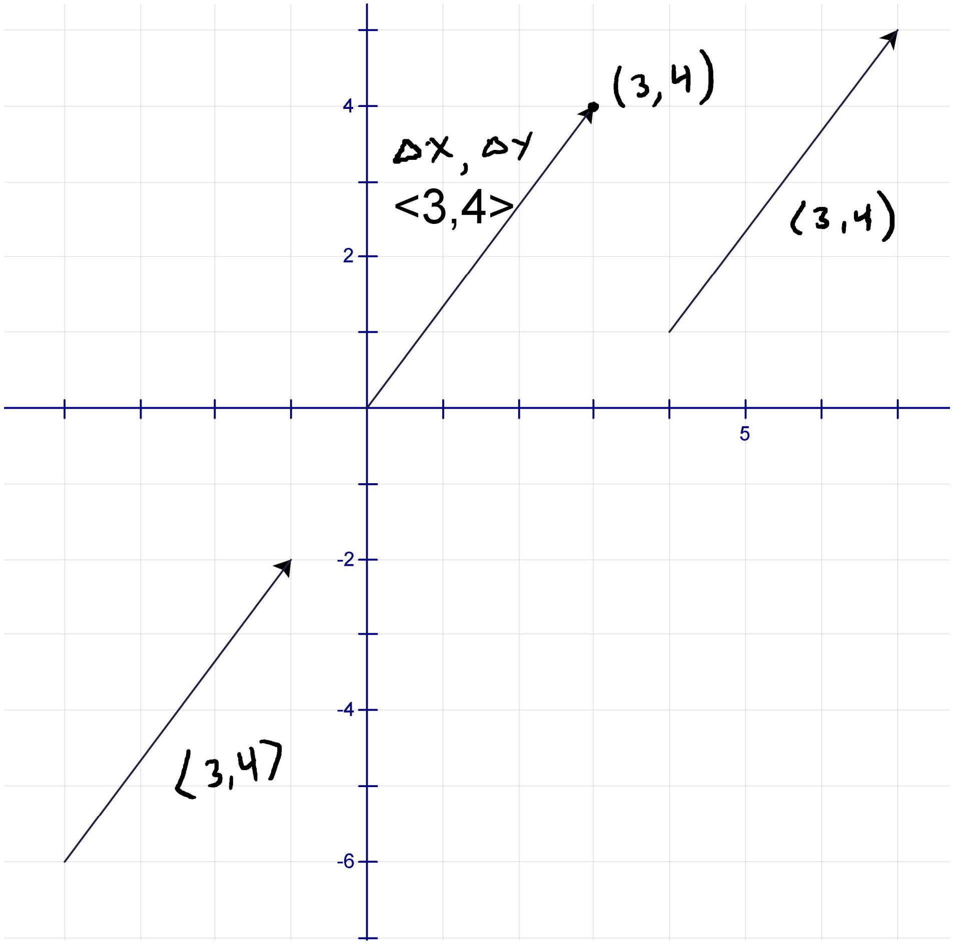
A is the <u>initial point</u>, *B* is the <u>terminal point</u>

Note that position was not used to determine a vector…



v and **u** are equivalent vectors

$$\mathbf{v} = \mathbf{u}$$



The vector from (0,0) to (3,4) can be written in <u>component form</u>

Because position doesn't matter, each of these vectors is equivalent

The vector representation that has an initial point at (0,0) is in <u>standard position</u>

Ex. Find the vector represented by the directed line segment with initial point $A(2,-3,4)$ and terminal point $B(-2,1,1)$.

$$\overrightarrow{AB} = \langle \underset{\Delta x}{-2-2},\ \underset{\Delta y}{1-(-3)},\ \underset{\Delta z}{1-4} \rangle = \langle -4, 4, -3 \rangle$$

Ex. Consider vector **v** from (0,0) to (3,2) and vector **u** from (1,2) to (4,4). Show that **u** and **v** are equivalent vectors.

$\vec{u} = \langle 4-1, 4-2 \rangle = \langle 3, 2 \rangle$

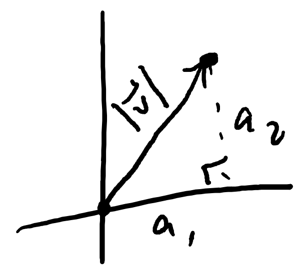
$\vec{v} = \langle 3-0, 2-0 \rangle = \langle 3, 2 \rangle$

<u>Def.</u> The <u>magnitude</u> of a vector is the distance between initial and terminal points.

$$\left|\vec{v}\right| \text{ or } \left\|\vec{v}\right\|$$

<u>Thm.</u> The length of vector $\vec{a} = \langle a_1, a_2 \rangle$ is

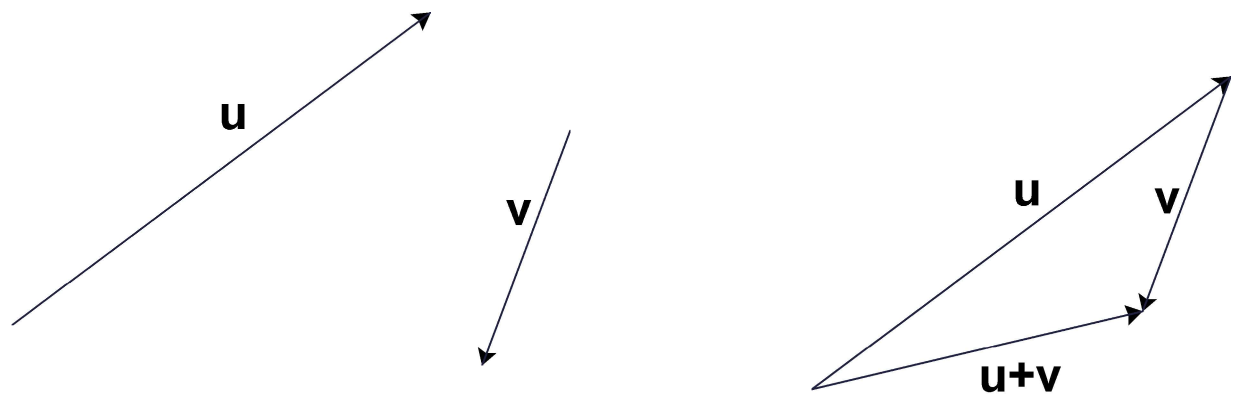
$$\left\|\vec{a}\right\| = \sqrt{a_1^{\ 2} + a_2^{\ 2}}$$



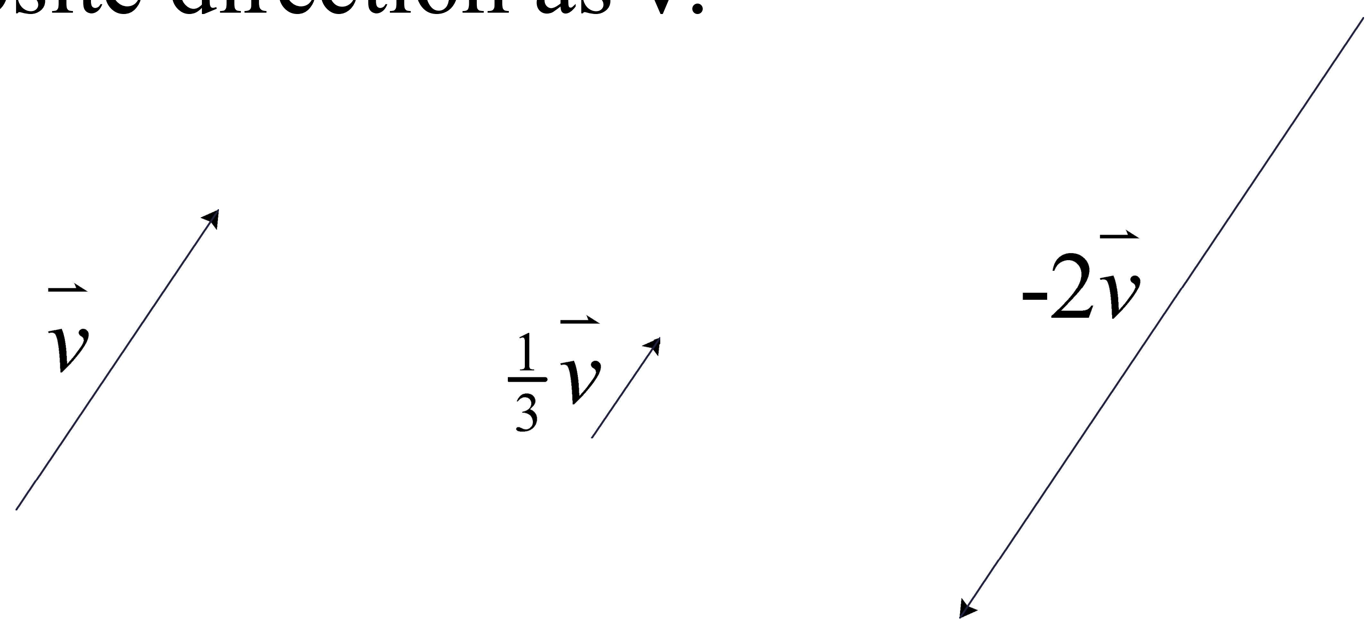
The length of vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is

$$\left\|\vec{a}\right\| = \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}}$$

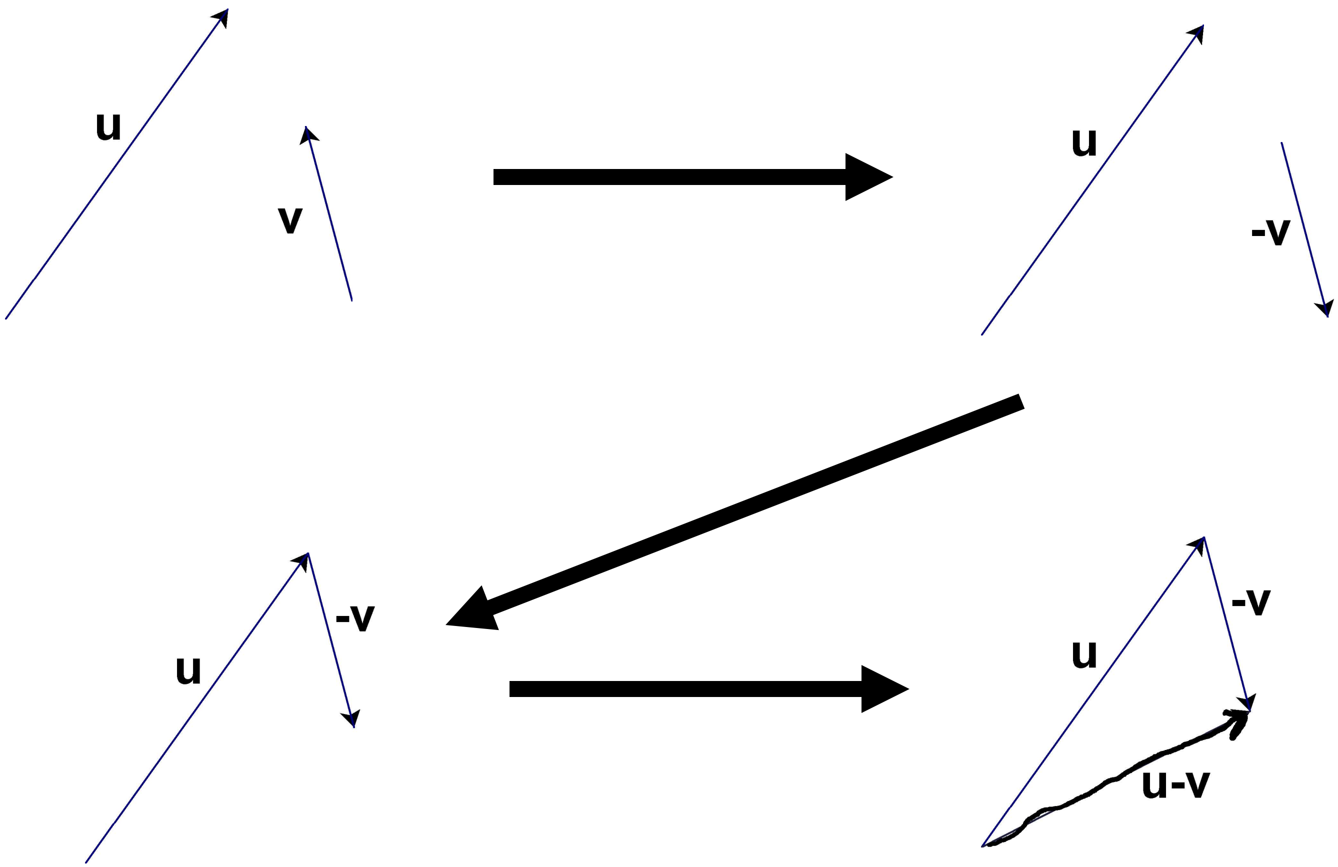
Def. If **u** and **v** are positioned so that the terminal point of **u** is at the initial point of **v**, then **u** + **v** is the vector with the initial point of **u** and the terminal point of **v**.



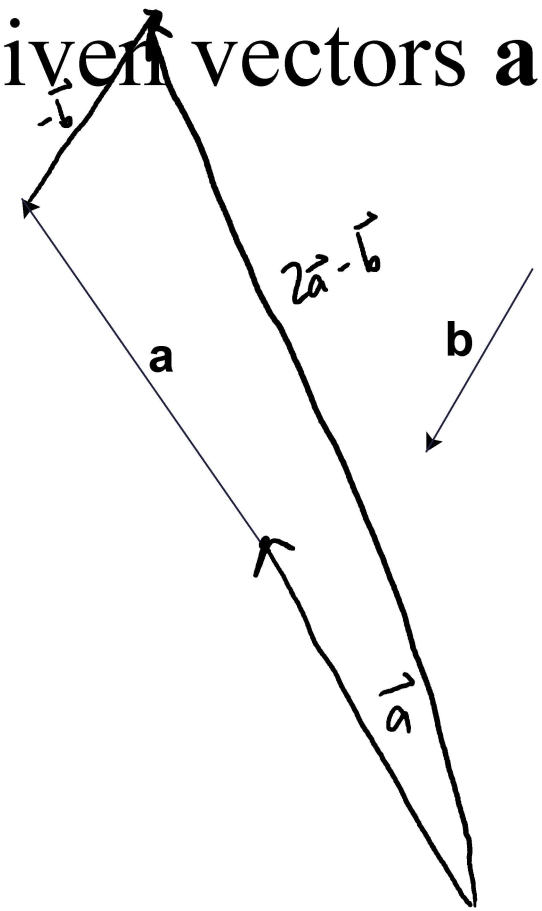
Def. If c is a scalar (non-vector) and **v** is a vector, then c**v** is the vector with the same direction as **v** that has length c times as long as **v**. If $c < 0$, then c**v** goes in the opposite direction as **v**.



$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (\mathbf{-v})$$



Ex. Given vectors **a** and **b**, draw 2**a** – **b**.



Adding and scalar multiplication numerically:

- When adding vectors in component form, add corresponding components
- When multiplying by a scalar, multiply each component by the scalar

→Note that we haven't talked about multiplying two vectors

Ex. Let $\vec{a} = \langle 4, 0, 3 \rangle$ and $\vec{b} = \langle -2, 1, 5 \rangle$, find

a) $\|\mathbf{a}\| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

b) $\mathbf{a} + \mathbf{b} = \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle$

c) $2\mathbf{a} - 5\mathbf{b} = \langle 8, 0, 6 \rangle - \langle -10, 5, 25 \rangle = \langle 18, -5, -19 \rangle$

Def. A unit vector is a vector whose length is 1.

The unit vector in the direction of **a** is

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$$

Ex. Find the unit vector in the direction of

$\vec{v} = \langle 2, -1, -2 \rangle$ $\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4 + 1 + 4} = 3$

$$\vec{u} = \frac{\langle 2, -1, -2 \rangle}{3} = \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle$$

Ex. Find a vector in the direction of $\vec{v} = \langle 3, 4, -2 \rangle$ that has magnitude 7.

$\|\vec{v}\| = \sqrt{3^2 + 4^2 + (-2)^2}$

$= \sqrt{9 + 16 + 4} = \sqrt{29}$

$\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \left\langle \dfrac{3}{\sqrt{29}}, \dfrac{4}{\sqrt{29}}, \dfrac{-2}{\sqrt{29}} \right\rangle$

$7\vec{u} = \left\langle \dfrac{21}{\sqrt{29}}, \dfrac{28}{\sqrt{29}}, \dfrac{-14}{\sqrt{29}} \right\rangle$

Note V_2 means the set of all vectors in 2-D, V_3 is the set of all vectors in 3-D

There are three basic vectors in V_3 that we should talk about:

$$\mathbf{i} = \langle 1,0,0 \rangle \qquad \mathbf{j} = \langle 0,1,0 \rangle \qquad \mathbf{k} = \langle 0,0,1 \rangle$$

→These are the standard basis vectors because they are unit vectors in each of the three dimensions

Every vector in V_3 can be described using **i**, **j**, and **k**

$$\langle 4,-2,1\rangle = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

Ex. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, find $2\mathbf{a} + 3\mathbf{b}$.

$$2(\hat{i} + 2\hat{j} - 3\hat{k}) + 3(4\hat{i} + 7\hat{k})$$

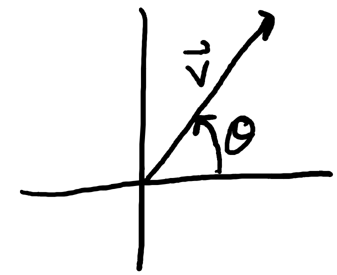
$$2\hat{i} + 4\hat{j} - 6\hat{k} + 12\hat{i} + 21\hat{k}$$

$$14\hat{i} + 4\hat{j} + 15\hat{k}$$

We can also find the vector if we know magnitude and angle:

$x = r\cos\theta$
$y = r\sin\theta$

$$\vec{v} = \|\vec{v}\|\cos\theta\,\hat{i} + \|\vec{v}\|\sin\theta\,\hat{j}$$



<u>Ex.</u> The magnitude of a vector is 3, and it forms an angle of $\frac{\pi}{6}$. Find the vector.

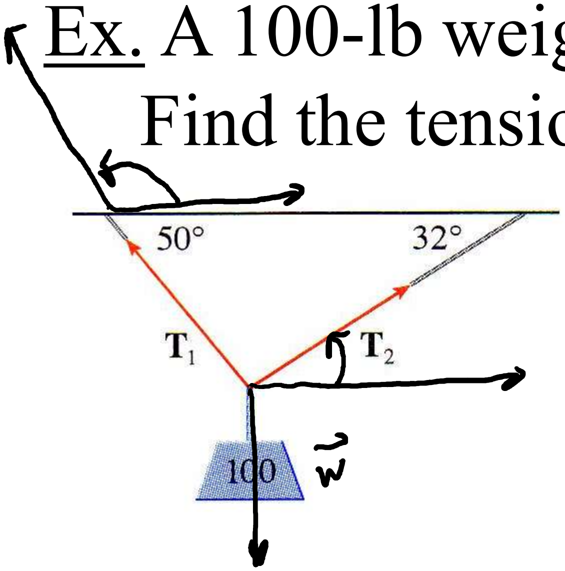
$$\vec{v} = \left\langle 3\cos\frac{\pi}{6},\ 3\sin\frac{\pi}{6} \right\rangle = \left\langle 3\cdot\frac{\sqrt{3}}{2},\ 3\cdot\frac{1}{2} \right\rangle$$

$$= \left\langle \frac{3\sqrt{3}}{2},\ \frac{3}{2} \right\rangle$$

In physics, vectors can be used to describe force:

If several forces act on an object, the resultant force is the sum of these forces.

Ex. A 100-lb weight hangs from two wires as shown. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires.



$\vec{T}_1 = \langle p\cos 130, p\sin 130\rangle$

$\vec{T}_2 = \langle q\cos 32, q\sin 32\rangle$

$W = \langle 0, -100\rangle$

$\|\vec{T}_1\| = p$

$\|\vec{T}_2\| = q$

$\vec{T}_1 + \vec{T}_2 + \vec{W} = \vec{0}$

$\langle p\cos 130 + q\cos 32, p\sin 130 + q\sin 32 - 100\rangle = \langle 0, 0\rangle$

$p\cos 130 + q\cos 32 = 0 \qquad p\sin 130 + q\sin 32 - 100 = 0$

$$p\cos 130 + q\cos 32 = 0 \qquad p\sin 130 + q\sin 32 - 100 = 0$$

$$q = \frac{-\cos 130}{\cos 32}p \quad \rightarrow \quad p\sin 130 + \left(\frac{-\cos 130}{\cos 32}p\right)\sin 32 - 100 = 0$$

$$p = 85.638$$

$$q = 64.910$$

$$\vec{T}_1 = \langle p\cos 130, p\sin 130\rangle = \langle -55.047, 65.603\rangle$$

$$\vec{T}_2 = \langle q\cos 32, q\sin 32\rangle = \langle 55.047, 34.397\rangle$$