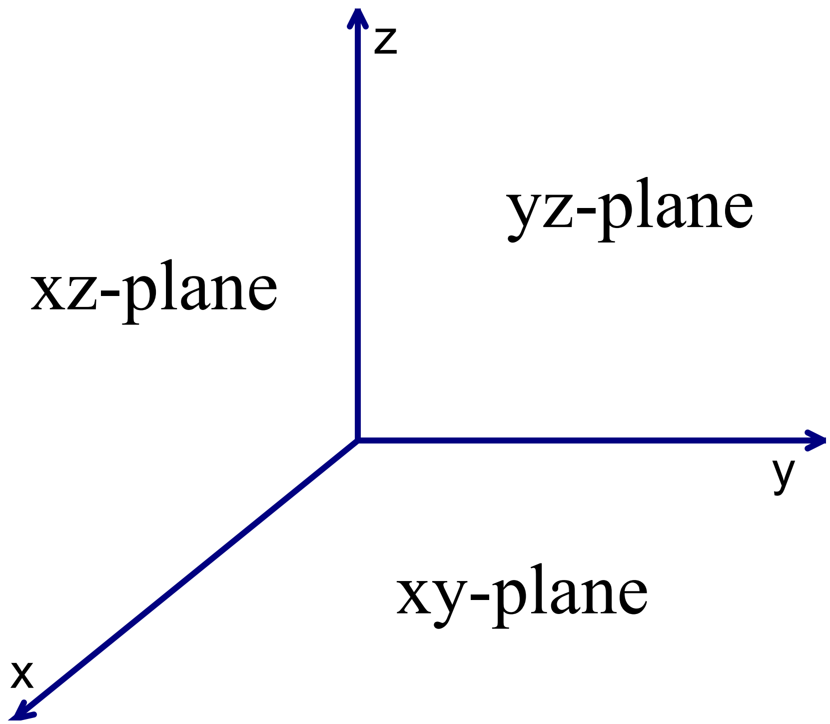
Graphing in 3-D



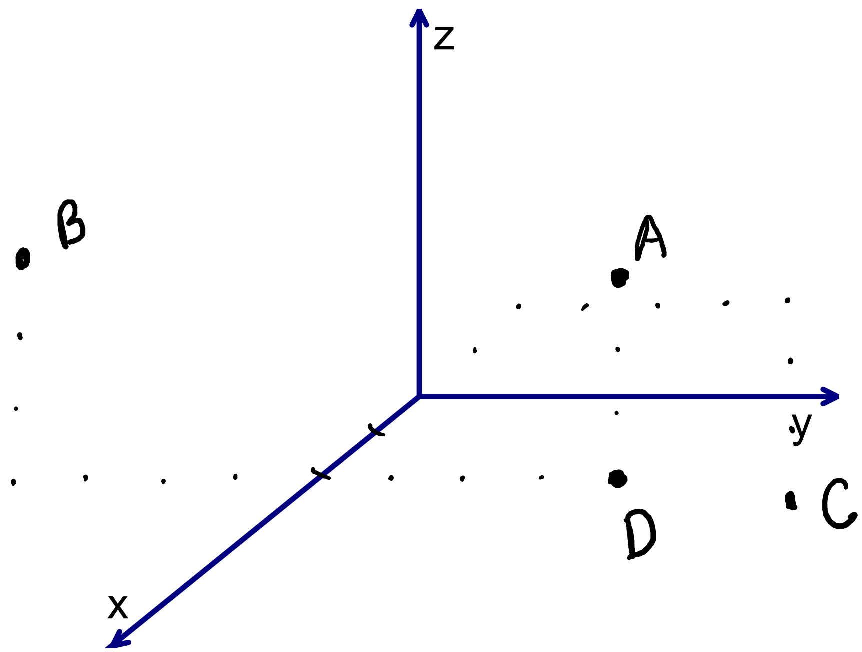
Graphing in 3-D means that we need 3 coordinates to define a point

$$(x,y,z)$$

These are the coordinate planes, and they divide space into 8 <u>octants</u>. The "1st octant" is where all 3 coordinates are positive

It is difficult to define a point in space when graphing on a plane.

<u>Ex.</u> Graph the points A(2,4,3), B(2,-4,3), C(-2,4,-3), and D(2,4,0).



This system for graphing in 3-D is called <u>rectangular</u>, or <u>Cartesian</u>.

Thm. Distance Formula in Three Dimensions

The distance between (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

Ex. Find the distance between $P(2,-1,7)$ and $Q(1,-3,5)$.

$$d=\sqrt{(2-1)^2+(-1--3)^2+(7-5)^2}$$

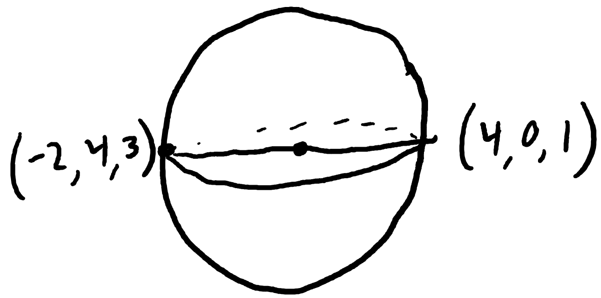
$$=\sqrt{1^2+2^2+2^2}=\sqrt{1+4+4}=\sqrt{9}=\boxed{3}$$

Thm. Equation of a Sphere

The sphere with center (h,k,l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

Ex. Find the equation of a sphere if a diameter has endpoints (-2,4,3) and (4,0,1).



center: $\left(\frac{-2+4}{2}, \frac{4+0}{2}, \frac{3+1}{2}\right) = \underline{(1, 2, 2)}$

$r = \sqrt{14}$

$$(x-1)^2+(y-2)^2+(z-2)^2=14$$

$$r = \sqrt{(4-1)^2+(0-2)^2+(1-2)^2}$$

$$= \sqrt{3^2+(-2)^2+(-1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

<u>Def.</u> Two vectors **u** and **v** are <u>parallel</u> if there is a scalar c such that $\mathbf{u} = c\mathbf{v}$.

<u>Ex.</u> Show that $\vec{u} = \langle 6, -4, 9 \rangle$ and $\vec{v} = \left\langle \frac{3}{4}, -\frac{1}{2}, \frac{9}{8} \right\rangle$ are parallel.

$$\vec{u} = \boxed{8}\,\vec{v}$$

$\vec{u} = \langle 6, -4 \rangle$ $\vec{v} = \left\langle \frac{3}{4}, -\frac{1}{2} \right\rangle$

$$m = \frac{\Delta y}{\Delta x} = \frac{-4}{6} = \frac{-1/2}{3/4}$$

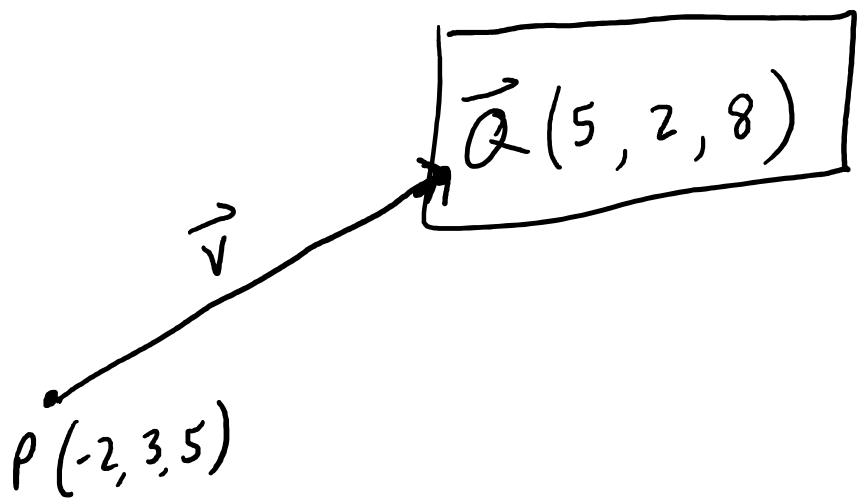
$$= -\frac{2}{3} = -\frac{2}{3}$$

$\vec{u} = \langle 6, -4 \rangle$ $\vec{v} = \langle 4, 6 \rangle$

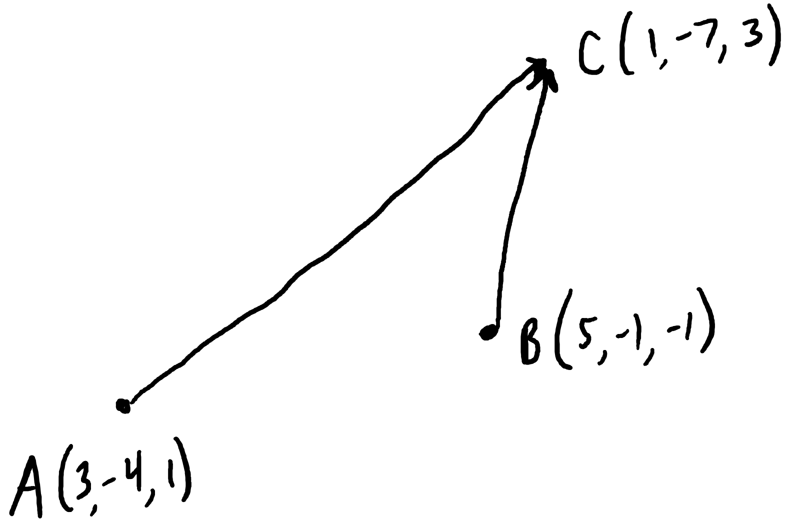
$m = \frac{-4}{6}$ $m = \frac{6}{4}$

<u>Ex.</u> Find the terminal point of $\vec{v} = \langle 7, -1, 3 \rangle$ if the initial point is P (-2,3,5).

$\Delta x, \Delta y, \Delta z$



Ex. Determine whether A (3,-4,1), B (5,-1,-1), and C (1,-7,3) are collinear.



$\overrightarrow{AC} = \langle -2, -3, 2 \rangle$

$\overrightarrow{BC} = \langle -4, -6, 4 \rangle$

$\overrightarrow{BC} = 2\overrightarrow{AC}$

parallel

$\Longrightarrow$ collinear

$\frac{2}{3}$ $\frac{4}{6}$

Dot Product

So far, we haven't talked about how to multiply two vectors…because there are two ways to "multiply" them.

Def. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

This is also called the scalar product, since the result is a scalar

Ex. Let $\vec{u} = \langle 2, -2 \rangle$ and $\vec{v} = \langle 5, 8 \rangle$, find:

a) $\mathbf{u} \cdot \mathbf{v} = 2(5) + (-2)(8) = 10 - 16 = -6$

b) $(\mathbf{v} \cdot \mathbf{v})\, \mathbf{u}$

$\vec{v} \cdot \vec{v} = 89$

$\|\vec{v}\| = \sqrt{5^2 + 8^2}$

$(5^2 + 8^2)\, \vec{u}$

$89\, \vec{u}$

$\langle 178, -178 \rangle$

Properties of the Dot Product

1) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4) $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5) $\mathbf{0} \cdot \mathbf{a} = 0$

$\vec{0} = \langle 0, 0 \rangle$

Dot product is used to find the angle between two vectors:

Thm. If θ is the angle between **a** and **b**, then

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

$\vec{a}\cdot\vec{b}=\underbrace{\|\vec{a}\|\,\|\vec{b}\|\cos 90}_{0}$

Ex. Find the angle between $\vec{a}=\langle 2,2,-1\rangle$ and $\vec{b}=\langle 5,-3,2\rangle$

$\vec{a}\cdot\vec{b}=\|\vec{a}\|\|\vec{b}\|\cos\theta$

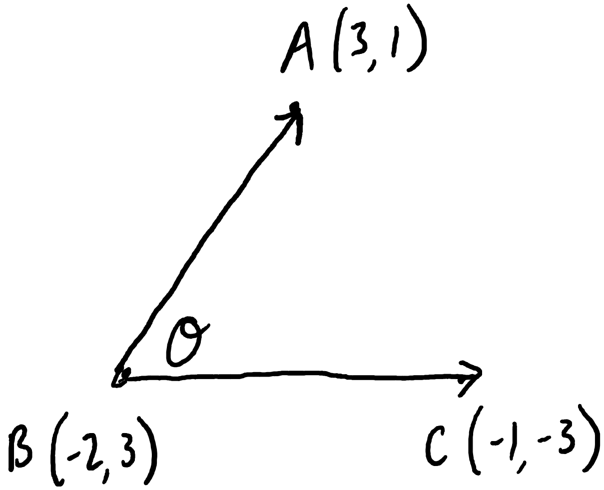
$10+(-6)+(-2)=\sqrt{2^2+2^2+(-1)^2}\,\sqrt{5^2+(-3)^2+2^2}\cos\theta$

$2=3\sqrt{38}\cos\theta$

$\cos\theta=\frac{2}{3\sqrt{38}}$

$\theta=1.462$

Ex. Consider the points $A(3,1)$, $B(-2,3)$ and $C(-1,-3)$. Find $m\angle ABC$.



$$\overrightarrow{BA} = \langle 5, -2 \rangle$$

$$\overrightarrow{BC} = \langle 1, -6 \rangle$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \|\overrightarrow{BA}\| \, \|\overrightarrow{BC}\| \cos\theta$$

$$5 + 12 = \sqrt{25+4} \, \sqrt{1+36} \cos\theta$$

$$17 = \sqrt{29}\sqrt{37} \cos\theta$$

$$\cos\theta = \frac{17}{\sqrt{29}\sqrt{37}}$$

$$\theta = 1.025$$

Thm. Two vectors **a** and **b** are orthogonal if
$\mathbf{a} \cdot \mathbf{b} = 0$.

Orthogonal = Perpendicular = Normal

Ex. Show that $\overset{\vec{u}\,''}{2\mathbf{i}} + 2\mathbf{j} - \mathbf{k}$ and $\overset{\vec{v}\,''}{5\mathbf{i}} - 4\mathbf{j} + 2\mathbf{k}$ are orthogonal

$\vec{u} \cdot \vec{v} = 10 + (-8) + (-2) = 0$ ✓

The <u>direction angles</u> of a vector **a** are the angles α, β, and γ that **a** makes with the positive x-, y-, and z-axes, respectively.

α

β

γ

Consider α…

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\hat{i} = \langle 1, 0, 0 \rangle$

It's the angle with the x-axis, so it's the angle with **i**.

$\vec{a} \cdot \hat{i} = \|\vec{a}\| \, \|\hat{i}\| \cos\alpha$

$a_1 = \|\vec{a}\| \cos\alpha \longrightarrow \cos\alpha = \dfrac{a_1}{\|a\|}$

$$\cos\alpha = \frac{a_1}{\left|\vec{a}\right|} \qquad \cos\beta = \frac{a_2}{\left|\vec{a}\right|} \qquad \cos\gamma = \frac{a_3}{\left|\vec{a}\right|}$$

These are called the <u>direction cosines</u> of **a**.

$$\frac{\vec{a}}{\left|\vec{a}\right|} = \left\langle \cos\alpha, \cos\beta, \cos\gamma \right\rangle$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Ex. Find the direction angles of $\vec{a} = \langle 1, 2, 3 \rangle$

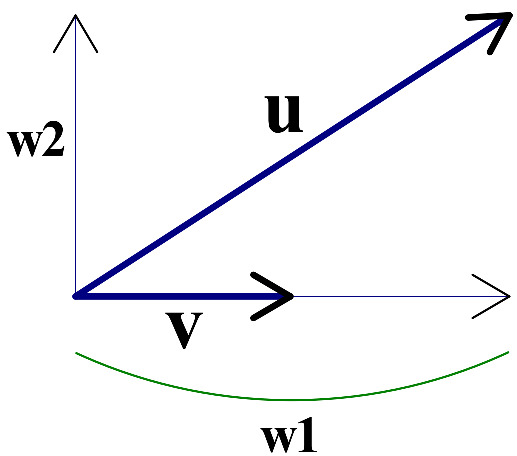
$\|\vec{a}\| = \sqrt{1+4+9} = \sqrt{14}$

$\cos\alpha = \frac{1}{\sqrt{14}} \longrightarrow \alpha = 1.300$

$\cos\beta = \frac{2}{\sqrt{14}} \longrightarrow \beta = 1.007$

$\cos\gamma = \frac{3}{\sqrt{14}} \longrightarrow \gamma = .641$

Def. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown.



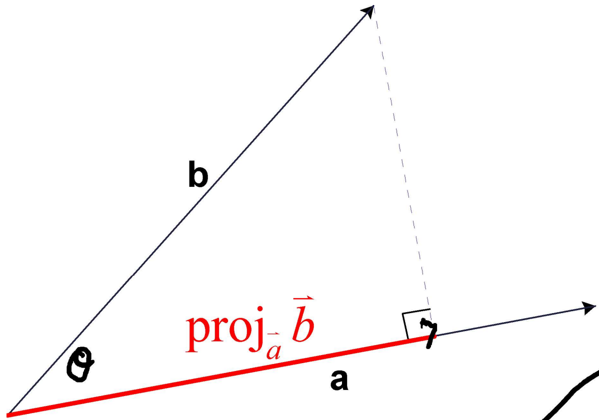
1) $\mathbf{w}_1$ is called the projection of $\mathbf{u}$ onto $\mathbf{v}$ or the vector component of $\mathbf{u}$ along $\mathbf{v}$, and is written $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$

2) $\mathbf{w}_2$ is called the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

→ Note $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$

Let’s find $\text{proj}_{\mathbf{a}}\mathbf{b}$

$x = \|\text{proj}_{a}\vec{b}\|$



$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\cancel{\|\vec{b}\|}\,\dfrac{x}{\cancel{\|\vec{b}\|}}$

$\cos\theta = \dfrac{x}{\|\vec{b}\|}$

$x = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$

$\text{proj}_{a}\vec{b} = \dfrac{\vec{a}}{\|\vec{a}\|}\cdot\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\left|\vec{v}\right|^2}\vec{v} \quad \text{or} \quad \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

Ex. Let **u** = 3**i** – 5**j** + 2**k** and **v** = 7**i** + **j** – 2**k**, find $\text{proj}_{\mathbf{v}}\mathbf{u}$ and the vector component of **u** orthogonal to **v**.

$$\text{proj}_{\vec v}\vec u = \frac{\vec u \cdot \vec v}{\vec v \cdot \vec v} \cdot \vec v = \frac{21-5-4}{49+1+4}\langle 7, 1, -2\rangle = \frac{12}{54}\langle 7,1,-2\rangle = \frac{2}{9}\langle 7, 1, -2\rangle$$

$$= \left\langle \frac{14}{9}, \frac{2}{9}, \frac{-4}{9} \right\rangle$$

comp. orthog. to $\vec v$ $= \vec u - \text{proj}_{\vec v}\vec u = \langle 3, -5, 2\rangle - \left\langle \frac{14}{9}, \frac{2}{9}, \frac{-4}{9}\right\rangle$

$\vec w_2$ $\qquad \vec w_1$

$$= \left\langle \frac{13}{9}, \frac{-47}{9}, \frac{22}{9} \right\rangle$$