Cross Product

Before discussing the second way to "multiply" vectors, we need to talk about matrices…

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A is

$$|A| = ad - bc \qquad \begin{vmatrix} 3 & 1 \\ 7 & 9 \end{vmatrix} = 27 - 7 = 20$$

To find the determinant of a 3 × 3 matrix, we will work along the top row.

Ex. $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix}$

$$= 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0-4) - 2(6-(-5)) + (-1)(12-0)$$

$$= -4 - 2(11) - 12 \quad = -4 - 22 - 12 = -38$$

<u>Def.</u> Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the <u>cross product</u> is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \square\,\hat{i} + \square\,\hat{j} + \square\,\hat{k}$$

This is also called the <u>vector product</u>, since the result is a vector

→This only makes sense in V_3

Ex. $\langle 1,3,4\rangle \times \langle 2,7,-5\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

$$= \hat{i}(-15-28) - \hat{j}(-5-8) + \hat{k}(7-6)$$

$$= -43\hat{i} + 13\hat{j} + \hat{k}$$

Thm. The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

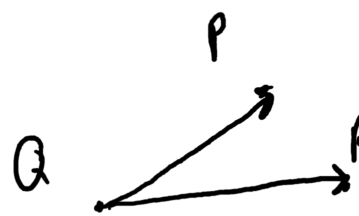
Thm. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

To find the orientation of $\mathbf{a} \times \mathbf{b}$, use your right hand:

$\mathbf{a}$ = index finger
$\mathbf{b}$ = three fingers
$\mathbf{a} \times \mathbf{b}$ = thumb

→Switching the order of the cross product reverses the orientation

Ex. Find a vector perpendicular to the plane that contains the points $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.



$\overrightarrow{QP} = \langle 3,-1,7 \rangle$

$\overrightarrow{QR} = \langle 3,-6,2 \rangle$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 7 \\ 3 & -6 & 2 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 7 \\ -6 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 7 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -1 \\ 3 & -6 \end{vmatrix}$$

$$= \hat{i}(-2-(-42)) - \hat{j}(6-21) + \hat{k}(-18-(-3))$$

$$= 40\hat{i} + 15\hat{j} - 15\hat{k}$$

$$\langle 8, 3, -3 \rangle$$

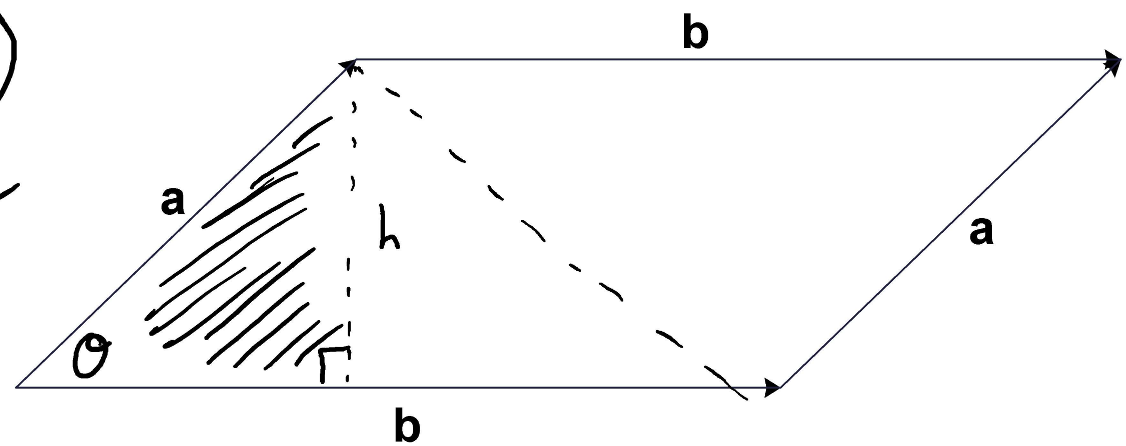
Thm. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where θ is the angle between **a** and **b**.

Cor. If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then **a** and **b** are parallel.

Thm. The magnitude of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram formed by **a** and **b**.

$\sin\theta = \dfrac{h}{\|\vec{a}\|}$

$h = \|\vec{a}\| \sin\theta$



$A = b \cdot h$

$A = \|\vec{b}\|\,\|\vec{a}\| \sin\theta$

$A = \|\vec{a} \times \vec{b}\|$

Ex. Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

$$\overrightarrow{QP} \times \overrightarrow{QR} = \langle -40, 15, -15 \rangle$$

$$\|\overrightarrow{QP} \times \overrightarrow{QR}\| = \sqrt{(-40)^2 + 15^2 + (-15)^2} = \sqrt{1600+225+225}$$

$$= \sqrt{2050} = 5\sqrt{82}$$

$$\boxed{A_\Delta = \frac{5\sqrt{82}}{2}}$$

2050
5 410
5 82

Properties of the Cross Product

1) $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

2) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

3) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

4) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

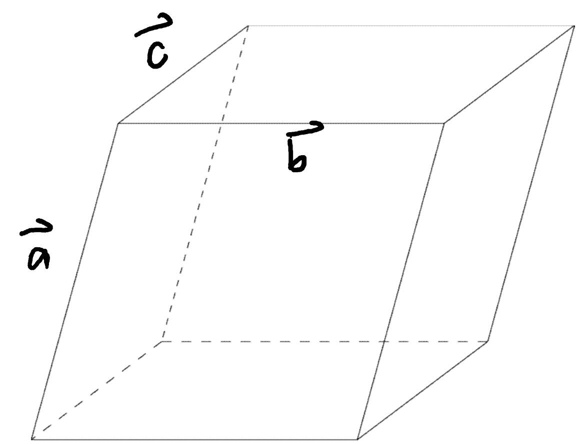
5) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

The quantity **a** · (**b** × **c**) is called the <u>scalar triple product</u>.

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

<u>Thm.</u> The volume of the parallelepiped determined by **a**, **b**, and **c** is

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$



<u>Ex.</u> Show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, and $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(-18 - (-36)) - 4(36) - 7(-18)$$

$$= 18 - 144 + 126 = 0$$

Lines and Planes

In three dimensions, we use vectors to indicate the direction of a line.

$\langle 7,6,0 \rangle$ as a direction vector would indicate that $\Delta x = 7,\ \Delta y = 6$, and $\Delta z = 0$ as points move along the line.

Of course, points also move along at multiples of these values, just like a slope of 2 doesn't mean that every point goes up 2 and over 1.

Thm. A line that contains the point $P(x_0, y_0, z_0)$ and direction vector $\langle a, b, c \rangle$ is defined by the equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

where t can be any real number

We call t the parameter of the equation, and these are called the parametric equations of the line.

The parametric equations of a line can be written in vector form:

$$\vec{r} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

r is the name of the line

Ex. Consider the line that passes through (5,1,3) and is parallel to $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

a) Find the vector and parametric equations of the line

$$\begin{cases} x = 5 + t \\ y = 1 + 4t \\ z = 3 - 2t \end{cases}$$

$$\vec{r} = \langle 5+t, 1+4t, 3-2t \rangle$$

b) Identify two other points on the line.

$t = 2$:

$x = 7$

$y = 9$

$z = -1$

$(7, 9, -1)$

$t = 5$:

$x = 10$

$y = 21$

$z = -7$

$(10, 21, -7)$

Another way to describe a line can be found by solving each parametric equation for *t*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These are called the symmetric equations of the line.

If one of the components of the direction vector is zero, we get equations like this:

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Ex. Find the parametric equations of the line that pass through $A(2,4,-3)$ and $B(3,-1,1)$. At what point does this line intersect the xy-plane?

$\overrightarrow{AB} = \langle 1, -5, 4 \rangle$

$$\boxed{\begin{cases} x = 2 + t \\ y = 4 - 5t \\ z = -3 + 4t \end{cases}}$$

$-3 + 4t = 0$

$t = \frac{3}{4}$

$x = 2 + \frac{3}{4} = \frac{11}{4}$

$y = 4 - 5\left(\frac{3}{4}\right) = \frac{1}{4}$

$z = 0$

$$\boxed{\left(\frac{11}{4}, \frac{1}{4}, 0\right)}$$

We can describe just a segment of a line by placing a restriction on the parameter, such as $0 \le t \le 1$

<u>Ex.</u> Determine if the lines are ~~parallel~~, intersecting, or skew.

$$\begin{cases} x = 1 + t \\ y = -2 + 3t \\ z = 4 - t = 4 - \frac{11}{5} = \boxed{\frac{9}{5}} \end{cases} \qquad \begin{cases} x = 2s \\ y = 3 + s \\ z = -3 + 4s = -3 + 4\left(\frac{8}{5}\right) = \boxed{\frac{17}{5}} \end{cases}$$

$\vec{u} = \langle 1, 3, -1 \rangle$ $\qquad$ $\vec{v} = \langle 2, 1, 4 \rangle$

$1 + t = 2s \longrightarrow t = 2s - 1$

$-2 + 3t = 3 + s$

$-2 + 3(2s - 1) = 3 + s$

$-2 + 6s - 3 = 3 + s$

$5s = 8$

$s = \frac{8}{5}$

$t = 2\left(\frac{8}{5}\right) - 1$

$t = \frac{11}{5}$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

Ex. Determine if the lines intersect, and find the angle of intersection.

$$x = 1 - 3t \qquad x = 1 + 3s$$
$$y = 1 + 4t \qquad y = 4 + 2s$$
$$z = 4 + 2t \qquad z = 1 - s$$

Now we want to find the equation of the plane that contains the point $P(x_0,y_0,z_0)$ and all vectors are normal to $\vec{n} = \langle a,b,c \rangle$

Let's choose an arbitrary point $Q(x,y,z)$ that lies on the plane.

- $\overrightarrow{PQ}$ is normal to $\vec{n}$
- $\overrightarrow{PQ} \cdot \vec{n} = 0$
- $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a,b,c \rangle = 0$
- $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Thm. The equation of the plane containing point $P(x_0,y_0,z_0)$ with normal vector $\vec{n}=\langle a,b,c\rangle$ is

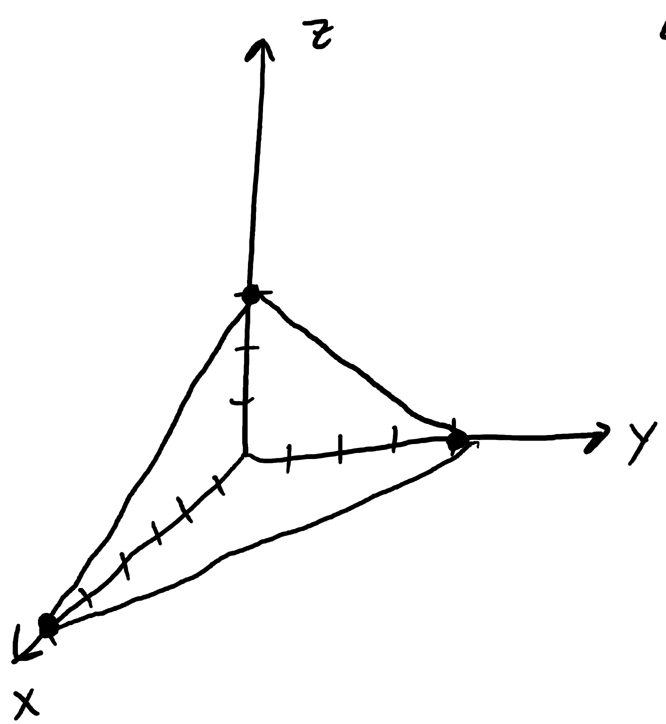
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Ex. Write the equation of the plane through point (2,4,-1) with normal vector $\vec{n} = \langle 2,3,4 \rangle$ and sketch a graph of the plane.

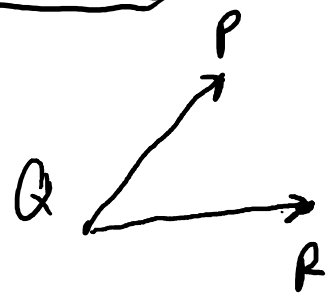
$$2(x-2)+3(y-4)+4(z+1)=0$$

$$2x-4+3y-12+4z+4=0$$

$$2x+3y+4z=12$$



Ex. Write the equation of the plane that contains the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.



$$\overrightarrow{QP} = \langle -2, 4, -4 \rangle$$
$$\overrightarrow{QR} = \langle 2, 3, -6 \rangle$$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & -4 \\ 2 & 3 & -6 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 4 & -4 \\ 3 & -6 \end{vmatrix} - \hat{j}\begin{vmatrix} -2 & -4 \\ 2 & -6 \end{vmatrix} + \hat{k}\begin{vmatrix} -2 & 4 \\ 2 & 3 \end{vmatrix}$$

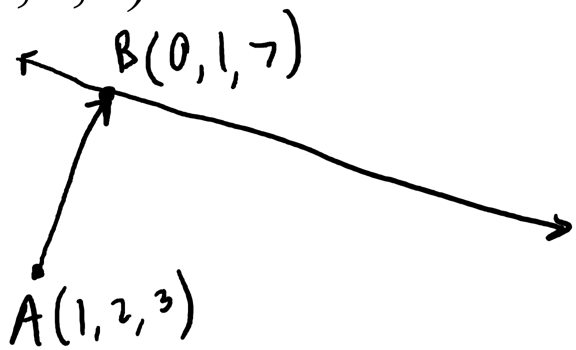
$$= \hat{i}(-24+12) - \hat{j}(12+8) + \hat{k}(-6-8)$$

$$= \langle -12, -20, -14 \rangle$$

$$\boxed{-12(x-1) - 20(y-3) - 14(z-2) = 0}$$

Ex. Write the equation of the plane that contains the point $A(1,2,3)$ and the line $x = 3t$, $y = 1 + t$, and $z = 7 + 4t$.

$\vec{v} = \langle 3, 1, 4 \rangle$



$\vec{AB} = \langle -1, -1, 4 \rangle$

$\vec{n} = \vec{AB} \times \vec{v} =$

Ex. Find the point at which the line $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$

$$4(2+3t)+5(-4t)-2(5+t)=18$$
$$8+12t-20t-10-2t=18$$
$$-10t=20$$
$$t=-2$$

$$x=2+3(-2) = -4$$
$$y=-4(-2) = 8$$
$$z=5+(-2) = 3$$

$$\boxed{(-4, 8, 3)}$$

$\vec{n}_1 = \langle 1, 1, 1 \rangle$

<u>Ex.</u> Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$, and then find the symmetric equations for the line of intersection.

$\vec{n}_2 = \langle 1, -2, 3 \rangle$

$$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \|\vec{n}_2\| \cos\theta$$

$$1 - 2 + 3 = \sqrt{1+1+1}\sqrt{1+4+9}\cos\theta$$

$$\cos\theta = \frac{2}{\sqrt{3}\sqrt{14}}$$

$$\theta = 1.257$$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= \hat{i}(3+2) - \hat{j}(3-1) + \hat{k}(-2-1)$$

$$= \langle 5, -2, -3 \rangle$$

$x + y + z = 1$

$x - 2y + 3z = 1$

find the pt. where $z = 0$

$$\begin{array}{r} x + y = 1 \\ -\quad x - 2y = 1 \\ \hline 3y = 0 \\ y = 0 \end{array}$$

$x = 1$

$(1, 0, 0)$

$$\boxed{\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}}$$

<u>Thm.</u> The distance between point $P(x_0,y_0,z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Ex. Find the distance between the planes

$10x + 2y - 2z = 5$ and $5x + y - z = 1$

$\vec{n}_1 = \langle 10, 2, -2 \rangle$

$\vec{n}_2 = \langle 5, 1, -1 \rangle$

$\left(0, 0, \frac{-5}{2}\right)$

$5x + y - z - 1 = 0$