Vector Functions

A <u>vector function</u> is a vector whose components are real-valued functions of a common variable (parameter), usually *t*.

→We've seen a vector function before… the parametric equations of a line

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

A general vector function would be

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\mathrm{i}} + g(t)\hat{\mathrm{j}} + h(t)\hat{\mathrm{k}}$$

These are vector functions because, when we evaluate $\mathbf{r}(t)$ at a value of t, the result is a vector.

The domain of $\mathbf{r}(t)$ is the intersection of the domains of the component functions.

Ex. Consider $\vec{r}(t)=\left\langle t^3, \ln(3-t), \sqrt{t}\right\rangle$, find

a) $\mathbf{r}(2)$ $= \langle 2^3, \ln(3-2), \sqrt{2} \rangle = \langle 8, 0, \sqrt{2} \rangle$

b) $|\mathbf{r}(t)|$ $= \sqrt{(t^3)^2 + [\ln(3-t)]^2 + (\sqrt{t})^2}$

c) the domain of $\mathbf{r}(t)$

$$\underbrace{\begin{array}{ccc} t^3 & \ln(3-t) & \sqrt{t} \\ \mathbb{R} & t<3 & t \geq 0 \end{array}}_{0 \leq t < 3}$$

Limits behave as we'd expect.

Ex. Find $\lim_{t\to 0}\vec{r}(t)$ where $\vec{r}(t)=\left\langle 1+t^3, te^{-t}, \frac{\sin t}{t}\right\rangle$

$= \langle 1+0^3, 0e^0, \frac{\sin 0}{0} \rangle = \langle 1, 0, 1 \rangle$

$\lim_{t\to 0} \frac{\sin t}{t} = 1$

Ex. Represent the curve as a vector function:

a) $y = x^2$

$\vec{r}(t) = \langle t, t^2 \rangle$

$\vec{r}(t) = \langle 3-t, (3-t)^2 \rangle$

$\vec{r}(t) = \cancel{\langle t^2, t^4 \rangle}$

b) $x^2 + y^2 = 25$

$\vec{r}(\theta) = \langle \cos\theta, \sin\theta \rangle$

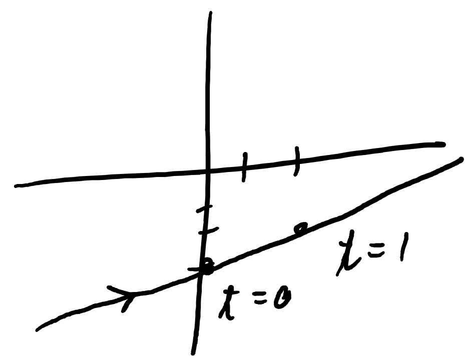
The graph of a vector function is called a <u>space curve</u>.

<u>Ex.</u> Sketch and indicate the orientation of the curve whose vector equation is
$\vec{r}(t) = \langle 2t, t-3 \rangle$ (first find rectangular)

$x = 2t \rightarrow t = \frac{x}{2}$

$y = t - 3$

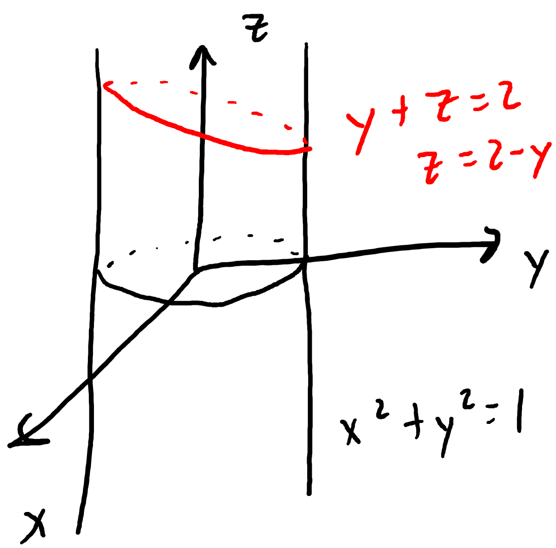
$\rightarrow y = \frac{x}{2} - 3$



Ex. Describe the curve whose vector equation is $\vec{r}(t) = \cos t\,\hat{\mathrm{i}} + \sin t\,\hat{\mathrm{j}} + t\,\hat{\mathrm{k}}$

spiral

Ex. Find the vector function for the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.



$\vec{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$

Derivatives and integrals behave as we'd like.

<u>Ex.</u> Consider $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin t \rangle$, find

a) $\mathbf{r}'(t) = \langle 3t^2, te^{-t}(-1) + e^{-t} \cdot 1, \cos t \rangle$

$= \langle 3t^2, e^{-t}(1-t), \cos t \rangle$

b) The unit tangent vector at the point where $t = 0$

$\vec{r}'(0) = \langle 0, 1, 1 \rangle$ $\quad \|\vec{r}'(0)\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$

$$\boxed{\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$$

Ex. Find the parametric equations of the line tangent to $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$ at $\left(0, 1, \frac{\pi}{2}\right)$

$t = \frac{\pi}{2}$

$\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle -2, 0, 1 \rangle$

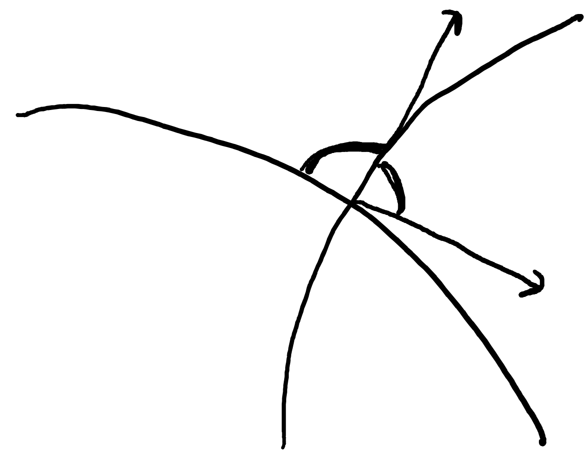
$$\begin{cases} x = 0 - 2t \\ y = 1 + 0t \\ z = \frac{\pi}{2} + 1t \end{cases}$$

$\|\vec{r}\| = \sqrt{4\sin^2 t + \cos^2 t + 1}$

Ex. The functions $\vec{r}_1(t) = \langle t^2, t, t^4 \rangle$ and $\vec{r}_2(t) = \langle e^t, e^{3t}, e^{2t} \rangle$ intersect at the point (1,1,1). Find the angle of intersection.

$t = 1$

$t = 0$



$\vec{r}_1\,'(t) = \langle 2t, 1, 4t^3 \rangle$

$\vec{r}_1\,'(1) = \langle 2, 1, 4 \rangle \longrightarrow \vec{a}$

$\vec{r}_2\,'(t) = \langle e^t, 3e^{3t}, 2e^{2t} \rangle$

$\vec{r}_2\,'(0) = \langle 1, 3, 2 \rangle \longrightarrow \vec{b}$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

$2 + 3 + 8 = \sqrt{2^2 + 1^2 + 4^2}\,\sqrt{1^2 + 3^2 + 2^2}\,\cos\theta$

$\cos\theta = \dfrac{13}{\sqrt{21}\,\sqrt{14}}$

A space curve given by the vector function $\mathbf{r}(t)$ is <u>smooth</u> if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.

<u>Ex.</u> Find all values of t where $\vec{r}(t) = \langle 1+t^3, t^2 \rangle$ is not smooth.

Look on p. 844 for the properties of vector function derivaties (sum, product, etc. rules)

Ex. Show that if $|\mathbf{r}(t)| = c$, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

$$\|\vec{r}\|^2 = c^2$$

$$\vec{r} \cdot \vec{r} = c^2$$

$$\vec{r} \cdot \vec{r}\,' + r \cdot \vec{r}\,' = 0$$

$$2(\vec{r} \cdot \vec{r}\,') = 0$$

$$\vec{r} \cdot \vec{r}\,' = 0$$

$\vec{r}$ is arthog. to $\vec{r}\,'$

$\underline{\text{Ex.}}$ $$\int_0^1 \left(\sqrt[3]{t}\,\hat{\mathrm{i}} + \tfrac{1}{1+t}\,\hat{\mathrm{j}} + e^{-t}\,\hat{\mathrm{k}} \right) dt$$

$$\frac{3}{4} \cdot t^{4/3} \Big|_0^1 \hat{i} + \ln|1+t| \Big|_0^1 \hat{j} + \left(-e^{-t}\right) \Big|_0^1 \hat{k}$$

$$\frac{3}{4}\hat{i} + (\ln 2)\hat{j} + \left(-e^{-1} + 1\right)\hat{k}$$

Ex. Find the antiderivative of

$\vec{r}'(t) = \left\langle \cos 2t, 2\sin t, \frac{1}{1+t^2} \right\rangle$ that satisfies

$\vec{r}(0) = \langle 3, -2, 1 \rangle$

$$\vec{r}(t) = \left\langle \tfrac{1}{2}\sin 2t + A,\ -2\cos t + B,\ \tan^{-1} t + C \right\rangle$$

$$\vec{r}(0) = \langle A,\ -2 + B,\ C \rangle = \langle 3, -2, 1 \rangle$$

$$A = 3 \qquad B = 0 \qquad C = 1$$

$$\vec{r}(t) = \left\langle \tfrac{1}{2}\sin 2t + 3,\ -2\cos t,\ \tan^{-1} t + 1 \right\rangle$$

If $\mathbf{r}(t)$ = displacement vector, then

$\mathbf{r}'(t)$ = velocity $\qquad \mathbf{v}(t)$

$\mathbf{r}''(t)$ = acceleration $\qquad \mathbf{a}(t)$

$\|\mathbf{r}'(t)\|$ = speed $\qquad v(t)$

Ex. Find the velocity, acceleration, and speed of a particle with position vector $\vec{r}(t)=\langle t^2, e^t, te^t \rangle$

$\vec{v}(t) = \langle 2t, e^t, te^t + e^t \cdot 1 \rangle = \langle 2t, e^t, e^t(t+1) \rangle$

$\vec{a}(t) = \langle 2, e^t, e^t \cdot 1 + (t+1)e^t \rangle = \langle 2, e^t, e^t(t+2) \rangle$

$v(t) = \sqrt{(2t)^2 + (e^t)^2 + [e^t(t+1)]^2}$

Ex. A particle starts with initial position $\vec{r}(0) = \langle 1,0,0 \rangle$ and initial velocity $\vec{v}(0) = \langle 1,-1,1 \rangle$. If its acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$, find the position and velocity functions.

$\vec{v}(t) = \langle 2t^2 + A,\ 3t^2 + B,\ t + C \rangle$

$\vec{v}(0) = \langle A, B, C \rangle = \langle 1, -1, 1 \rangle \rightarrow A = 1,\ B = -1,\ C = 1$

$\vec{v}(t) = \langle 2t^2 + 1,\ 3t^2 - 1,\ t + 1 \rangle$

$\vec{r}(t) = \langle \frac{2}{3}t^3 + t + D,\ t^3 - t + E,\ \frac{1}{2}t^2 + t + F \rangle$

$\vec{r}(0) = \langle D, E, F \rangle = \langle 1, 0, 0 \rangle \rightarrow D = 1,\ E = 0,\ F = 0$

$\vec{r}(t) = \langle \frac{2}{3}t^3 + t + 1,\ t^3 - t,\ \frac{1}{2}t^2 + t \rangle$

Newton’s Second Law of Motion:

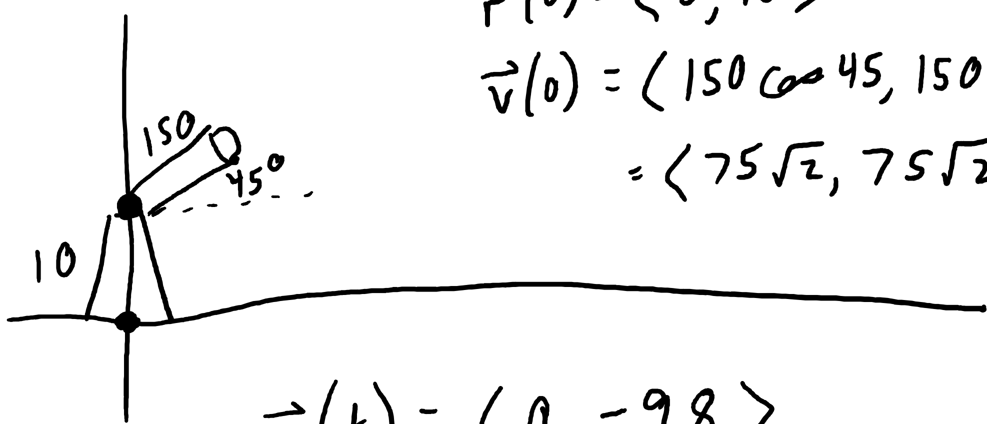
$$\mathbf{F}(t) = m\mathbf{a}(t)$$

Ex. A projectile is fired with muzzle speed 150 m/s and an angle of 45º from a position 10m above the ground. Where does the projectile hit the ground, and with what speed?

$\vec{r}(0) = \langle 0, 10 \rangle$

$\vec{v}(0) = \langle 150 \cos 45, 150 \sin 45 \rangle$

$= \langle 75\sqrt{2}, 75\sqrt{2} \rangle$



$\vec{a}(t) = \langle 0, -9.8 \rangle$

$\vec{v}(t) = \langle A, -9.8t + B \rangle$

$\vec{v}(0) = \langle A, B \rangle = \langle 75\sqrt{2}, 75\sqrt{2} \rangle$

$\vec{v}(t) = \langle 75\sqrt{2}, -9.8t + 75\sqrt{2} \rangle$

$\vec{r}(t) = \langle 75\sqrt{2}t + C, -4.9t^2 + 75\sqrt{2}t + D \rangle$

$\vec{r}(0) = \langle C, D \rangle = \langle 0, 10 \rangle$

$\vec{r}(t) = \langle 75\sqrt{2}t, -4.9t^2 + 75\sqrt{2}t + 10 \rangle$

$\vec{v}(t) = \langle 75\sqrt{2}, -9.8t + 75\sqrt{2} \rangle$ $\vec{r}(t) = \langle 75\sqrt{2}\,t, -4.9t^2 + 75\sqrt{2}\,t + 10 \rangle$

When hit?

$$-4.9t^2 + 75\sqrt{2}\,t + 10 = 0$$

$$t = 21.74 \text{ sec.}$$

Where hit?

$$75\sqrt{2}(21.74) = 2305.88 \text{ m}$$

Speed? $\|\vec{v}(21.74)\| = \sqrt{(75\sqrt{2})^2 + (-9.8(21.74) + 75\sqrt{2})^2}$

$$= 150.65 \text{ m/s.}$$