Tangent and Normal Vectors

Let **r**(*t*) be a smooth curve, then the unit tangent vector, **T**(*t*), is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}$$

Ex. Let $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j}$, find the unit tangent vector to the curve at $t = 1$.

$$\vec{r}\,' = \langle 1, 2t \rangle$$

$$\|\vec{r}\,'\| = \sqrt{1 + 4t^2}$$

$$\vec{T}(t) = \frac{\langle 1, 2t \rangle}{\sqrt{1 + 4t^2}} = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle$$

$$\vec{T}(1) = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

Ex. Find $\mathbf{T}(t)$ and the parametric equations of the tangent line to $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\,\mathbf{k}$ at $t = \frac{\pi}{4}$.

$\vec{r}\,' = \langle -2\sin t, 2\cos t, 1 \rangle$

$\vec{r}\,'\left(\frac{\pi}{4}\right) = \langle -\sqrt{2}, \sqrt{2}, 1 \rangle$

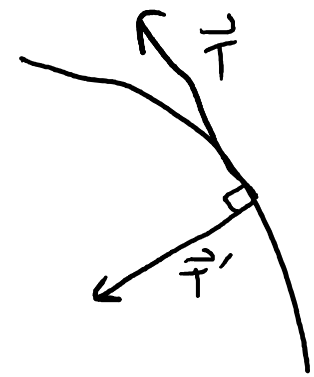
$\left\| \vec{r}\,'\left(\frac{\pi}{4}\right) \right\| = \sqrt{2+2+1} = \sqrt{5}$

$\vec{T}\left(\frac{\pi}{4}\right) = \left\langle \frac{-\sqrt{2}}{\sqrt{5}}, \frac{\sqrt{2}}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$

$\vec{r}\left(\frac{\pi}{4}\right) = \langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \rangle$

$$\begin{cases} x = \sqrt{2} - \sqrt{2}\,t \\ y = \sqrt{2} + \sqrt{2}\,t \\ z = \frac{\pi}{4} + t \end{cases}$$

We know that $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}$



→ $|\mathbf{T}(t)| = 1$ for all t.

→ We saw last class that if $|\mathbf{u}| = c$, then $\mathbf{u} \cdot \mathbf{u}' = 0$.

→ So $\mathbf{T} \cdot \mathbf{T}' = 0$ for all t.

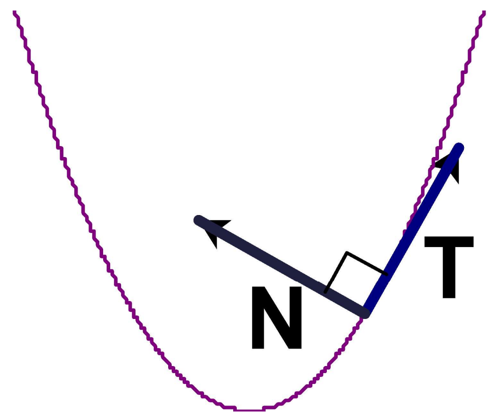
→ $\mathbf{T}'$ is orthogonal to $\mathbf{T}$ for all t.

We are going to define the <u>principal unit normal vector</u> as

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$$

T shows the direction that the curve is going

N shows the direction that the curve is turning



<u>Ex.</u> Find the principal unit normal vector of

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\vec{r}\,' = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}\,'\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\vec{T} = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{T}\,' = \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \right\rangle$$

$$\|\vec{T}\,'\| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\boxed{\vec{N} = \langle -\cos t, -\sin t, 0 \rangle}$$

For plane curves, if $\mathbf{T}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{N}(t)$ is either

$$-y(t)\mathbf{i} + x(t)\mathbf{j} \quad \text{or} \quad y(t)\mathbf{i} - x(t)\mathbf{j}.$$

→Both are normal, but only one is the principal normal.

Back to velocity and acceleration:

→ $|\mathbf{v}(t)| = \text{constant}$

→ $\mathbf{v} \cdot \mathbf{v} = c$

→ $\mathbf{v} \cdot \mathbf{a} = 0$

So if velocity is constant, then velocity is orthogonal to acceleration

→ This is not necessarily true for variable velocity

Because **T**(t) and **N**(t) are orthogonal, they define a plane.

<u>Thm.</u> If **r**(t) is a smooth function for position and if **N**(t) exists, then the acceleration vector **a**(t) lies in the plane determined by **T**(t) and **N**(t).

→This means that **a**(t) is a linear combination of **T**(t) and **N**(t):

$\vec{a} = \square\vec{T} + \triangle\vec{N}$

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

const.

$$\begin{pmatrix} \text{tangential} \\ \text{component of } \vec{a} \end{pmatrix} = a_T = \vec{a} \cdot \vec{T} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\left|\vec{r}'(t)\right|}$$

$$\begin{pmatrix} \text{normal} \\ \text{component of } \vec{a} \end{pmatrix} = a_N = \vec{a} \cdot \vec{N} = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|}$$

Ex. Find the tangential and normal components of acceleration for the position function $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$

$\vec{r}\,' = \langle 2t, 2t, 3t^2 \rangle$

$\vec{r}\,'' = \langle 2, 2, 6t \rangle$

$\|\vec{r}\,'\| = \sqrt{4t^2+4t^2+9t^4}$

$= \sqrt{8t^2+9t^4}$

$\vec{r}\,' \cdot \vec{r}\,'' = 4t + 4t + 18t^3 = 8t + 18t^3$

$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix}$

$= \hat{i} \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - \hat{j} \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} + \hat{k} \begin{vmatrix} 2t & 2t \\ 2 & 2 \end{vmatrix}$

$= \hat{i}\,(12t^2 - 6t^2) - \hat{j}\,(12t^2 - 6t^2) + \hat{k}\,(4t - 4t) = \langle 6t^2, -6t^2, 0 \rangle$

$\|\vec{r}\,' \times \vec{r}\,''\| = \sqrt{36t^4 + 36t^4 + 0} = 6t^2\sqrt{2}$

$a_T = \dfrac{8t + 18t^3}{\sqrt{8t^2+9t^4}}$ $\qquad$ $a_N = \dfrac{6t^2\sqrt{2}}{\sqrt{8t^2+9t^4}}$

<u>Ex.</u> Let $\vec{r}(t) = \left\langle t, t^2, \frac{t^2}{2} \right\rangle$, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, a_T, and a_N at $t = 1$.

[Use $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ to find $\mathbf{N}$.]

$\vec{r}\,' = \langle 1, 2t, t \rangle \longrightarrow \vec{r}\,'(1) = \langle 1, 2, 1 \rangle$

$\vec{r}\,'' = \langle 0, 2, 1 \rangle \longrightarrow \vec{r}\,''(1) = \langle 0, 2, 1 \rangle$

$\|\vec{r}\,'\| = \sqrt{1 + 4t^2 + t^2}$

$= \sqrt{1 + 5t^2} \longrightarrow \|\vec{r}\,'(1)\| = \sqrt{6}$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= \langle 0, -1, 2 \rangle$$

$\|\vec{r}\,' \times \vec{r}\,''\| = \sqrt{0 + 1 + 4} = \sqrt{5}$

$\vec{T}(1) = \dfrac{\langle 1, 2, 1 \rangle}{\sqrt{6}} = \left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}} \right\rangle$

$a_T(1) = \vec{a} \cdot \vec{T} = \langle 0, 2, 1 \rangle \cdot \left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}} \right\rangle = \dfrac{4}{\sqrt{6}} + \dfrac{1}{\sqrt{6}} = \dfrac{5}{\sqrt{6}}$

$a_N(1) = \dfrac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|} = \dfrac{\sqrt{5}}{\sqrt{6}}$

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$\vec{a}(1) = a_T(1)\vec{T}(1) + a_N(1)\vec{N}(1)$$

$$\langle 0, 2, 1 \rangle = \frac{5}{\sqrt{6}} \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle + \frac{\sqrt{5}}{\sqrt{6}} \vec{N}$$

$$\langle 0, 2, 1 \rangle = \left\langle \frac{5}{6}, \frac{5}{3}, \frac{5}{6} \right\rangle + \frac{\sqrt{5}}{\sqrt{6}} \vec{N}$$

$$\left\langle -\frac{5}{6}, \frac{1}{3}, \frac{1}{6} \right\rangle = \frac{\sqrt{5}}{\sqrt{6}} \vec{N}$$

$$\boxed{\frac{\sqrt{6}}{\sqrt{5}} \left\langle -\frac{5}{6}, \frac{1}{3}, \frac{1}{6} \right\rangle = \vec{N}}$$