Arc Length and Curvature

When learning arc length with parametric functions, we found that the length of a curve defined by $x = f(t)$ and $y = g(t)$ over the interval $a \leq t \leq b$ was

$$s = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt$$

We use the same formula for the length of the curve given by $\vec{r}(t) = f(t)\,\hat{\mathrm{i}} + g(t)\,\hat{\mathrm{j}}$ on the interval $a \leq t \leq b$.

$$s = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2 + \left[h'(t)\right]^2}\,dt$$

If $\vec{r}(t) = \left\langle f(t), g(t), h(t) \right\rangle$, we could also write

$\vec{r}\,' = \langle f', g', h' \rangle$

$$s = \int_a^b \left| \vec{r}\,'(t) \right| dt$$

Ex. Find the length of the curve

$t=0$ $\quad$ $t=2\pi$

$\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ from $(1,0,0)$ to $(1,0,2\pi)$

$$S = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt = \int_0^{2\pi} \sqrt{2}\, dt$$

$$= t\sqrt{2}\Big|_0^{2\pi}$$

$$= \boxed{2\pi\sqrt{2}}$$

These vector functions are the same:

$$\langle 1,1,1\rangle \qquad \langle 2,4,8\rangle$$

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \le t \le 2$$

$$t = e^u$$

$$\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \le u \le \ln 2$$

$$(2,4,8) \qquad (1,1,1)$$

$$\vec{r}_3(w) = \langle (1-w), (1-w)^2, (1-w)^3 \rangle \qquad -1 \le w \le 0$$

$$\vec{r}_4(t) = \left\langle \tfrac{t}{4}, \left(\tfrac{t}{4}\right)^2, \left(\tfrac{t}{4}\right)^3 \right\rangle \qquad 4 \le t \le 8$$

They are just reparameterizations of the same curve.

The arc length function is defined as

$$s(t)=\int_a^t \sqrt{\left[f'(u)\right]^2+\left[g'(u)\right]^2+\left[h'(u)\right]^2}\,du$$

We will see how to reparameterize a curve using s.

Ex. Reparameterize the curve

$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to the arc length measured from (1,0,0). $\rightarrow t=0$

$$s(t) = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2}\, du = \int_0^t \sqrt{2}\, du = u\sqrt{2}\Big|_0^t$$

$$= t\sqrt{2}$$

$$s = t\sqrt{2}$$

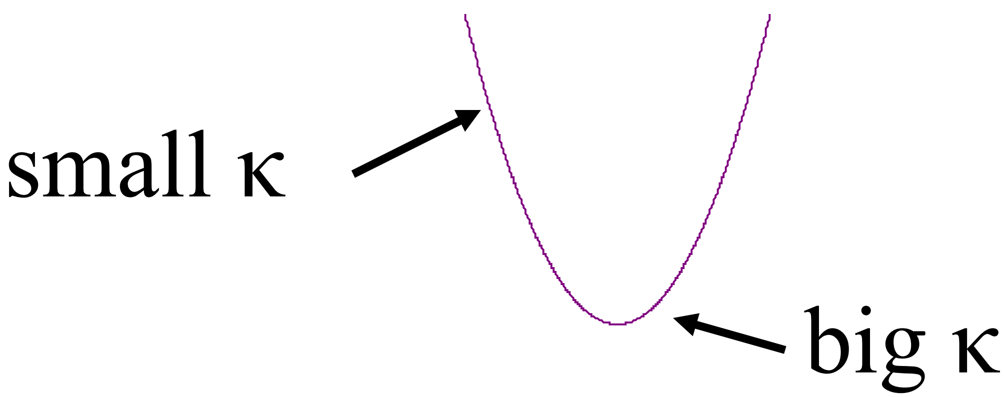
$$t = \frac{s}{\sqrt{2}}$$

$$\vec{r}(s) = \left\langle \cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ \frac{s}{\sqrt{2}} \right\rangle$$

Remember the unit tangent vector of $\mathbf{r}(t)$ is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}$$

The <u>curvature</u>, κ, of a curve describes how sharply it bends.



Thm. The curvature of a curve is

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

where **T** is the unit tangent vector and s is the arc length parameter.

$$\| \vec{T}'(s) \|$$

<u>Ex.</u> Find the curvature of $\vec{r}(s) = \left\langle 3\cos\frac{3}{s}, 3\sin\frac{3}{s} \right\rangle$

$$\vec{r}\,'(s) = \left\langle -3\sin\frac{3}{s}\cdot\frac{-3}{s^2}, 3\cos\frac{3}{s}\cdot\frac{-3}{s^2} \right\rangle$$

$$\|\vec{r}\,'(s)\| = \sqrt{\frac{81}{s^4}\sin^2\frac{3}{s} + \frac{81}{s^4}\cos^2\frac{3}{s}} = \sqrt{\frac{81}{s^4}}$$

$$= \frac{9}{s^2}$$

$$\vec{T}(s) = \frac{\left\langle \frac{9}{s^2}\sin\frac{3}{s}, \frac{-9}{s^2}\cos\frac{3}{s} \right\rangle}{9/s^2}$$

$$= \left\langle \sin\frac{3}{s}, -\cos\frac{3}{s} \right\rangle$$

$$\vec{T}\,'(s) = \left\langle \cos\frac{3}{s}\cdot\frac{-3}{s^2}, \sin\frac{3}{s}\cdot\frac{-3}{s^2} \right\rangle$$

$$K = \|\vec{T}\,'(s)\| = \sqrt{\frac{9}{s^4}\cos^2\frac{3}{s} + \frac{9}{s^4}\sin\frac{3}{s}} = \sqrt{\frac{9}{s^4}} = \boxed{\frac{3}{s^2}}$$

An easier equation would be

$$\kappa = \frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$$

Ex. Find the curvature of the circle with radius a.

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\vec{r}' = \langle -a\sin t, a\cos t\rangle$

$\|\vec{r}'\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

$\vec{T} = \dfrac{\langle -a\sin t, a\cos t\rangle}{a} = \langle -\sin t, \cos t\rangle$

$\vec{T}' = \langle -\cos t, -\sin t\rangle$

$\|\vec{T}'\| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$K = \dfrac{1}{a}$

We can also find curvature using

$$\kappa = \frac{\left| \vec{r}'(t) \times \vec{r}''(t) \right|}{\left| \vec{r}'(t) \right|^3}$$

Ex. Find the curvature of the space curve

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at any point t.

$\vec{r}\,' = \langle 1, 2t, 3t^2 \rangle \qquad \|\vec{r}\,'\| = \sqrt{1 + 4t^2 + 9t^4}$

$\vec{r}\,'' = \langle 0, 2, 6t \rangle$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \hat{i}(12t^2 - 6t^2) - \hat{j}(6t - 0) + \hat{k}(2 - 0)$$

$$= \langle 6t^2, -6t, 2 \rangle$$

$$\|\vec{r}\,' \times \vec{r}\,''\| = \sqrt{36t^4 + 36t^2 + 4}$$

$$K = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

If we're looking for the curvature of a plane curve $y = f(x)$, there's yet another formula:

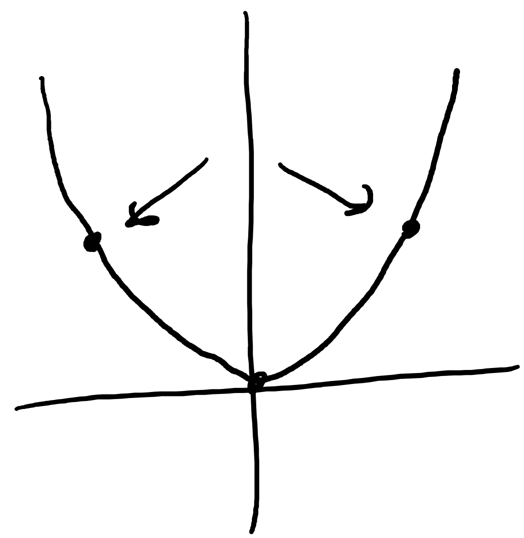
$$\kappa = \frac{\left|f''(x)\right|}{\left[1+\left(f'(x)\right)^2\right]^{3/2}}$$

Ex. Find the curvature of the $y = x^2$ at the points (0,0), (1,1), and (-1,1).

$y' = 2x$

$y'' = 2$

$$K = \frac{|2|}{(1 + (2x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$



$(0,0)$: $K = 2$

$(1,1)$: $K = \frac{2}{5^{3/2}}$

$(-1,1)$: $K = \frac{2}{5^{3/2}}$

p. 877 has a summary of all the formulas from this chapter