Differentials and Error

In one variable, if $y = f(x)$, the equation $dy = f'(x)dx$ is called the <u>differential</u>

In two variables, if $z = f(x,y)$, the equation becomes $dz = f_x dx + f_y dy$

→ This can be called the "total differential", and we treat dx like Δx.

Ex. Let $z = x^2 + 3xy - y^2$. Find the total differential and compare the values of dz and Δz as x changes from 2 to 2.05 and y changes from 3 to 2.96. $(2,3) \to (2.05, 2.96)$

$z_x = 2x + 3y$

$z_x(2,3) = 4 + 9 = 13$

$z_y = 3x - 2y$

$z_y(2,3) = 6 - 6 = 0$

$dz = z_x\,dx + z_y\,dy$

$= 13(.05) + 0(-.04)$

$= .65$

$\Delta z = z(2.05, 2.96) - z(2,3)$

$= .6449$

Ex. Approximate $(2.03)^2(1 + 8.9)^3 - 2^2(1 + 9)^3$

$f(2.03, 8.9) - f(2, 9)$

$(2,9) \to (2.03, 8.9)$

$f(x,y) = x^2(1+y)^3$

$f_x = 2x(1+y)^3$

$f_y = 3x^2(1+y)^2$

$f_x(2,9) = 4(10)^3 = 4000$

$f_y(2,9) = 12 \cdot 10^2 = 1200$

$dz = f_x\,dx + f_y\,dy$

$= 4000(.03) + 1200(-.1)$

$= 120 - 120 = 0$

Ex. The base radius and height of a right circular cone are measured as 10cm and 25cm, respectively, with a possible error in measurement of as much as 0.1 each. Estimate the maximum error in the calculated volume of the cone.

$$V = f(r,h) = \frac{1}{3}\pi r^2 h$$

at $(10,25)$, $V = \frac{1}{3}\pi(10)^2 \cdot 25 = \frac{2500\pi}{3}$

what is Δf as $(10,25) \to (10.1, 25.1)$

$$dV = f_r\,dr + f_h\,dh$$

$$= \frac{500\pi}{3}(.1) + \frac{100\pi}{3}(.1)$$

$$= \frac{50\pi}{3} + \frac{10\pi}{3} = 20\pi$$

$f_r = \frac{2}{3}\pi r h$ | $f_h = \frac{1}{3}\pi r^2$

at $(10,25)$

$= \frac{2}{3}\pi(10)(25) = \frac{500\pi}{3}$ | $= \frac{1}{3}\pi(10)^2 = \frac{100\pi}{3}$

Chain Rule

In Calculus I, we learned the chain rule:

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)g'\left(x\right)$$

Another way to write this would be to assume that $y = f(x)$ and $x = g(t)$:

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

In multivariable, this second method is used.

Let $w = f(x,y)$, with $x = g(t)$ and $y = h(t)$:

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$

<u>Ex.</u> Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$.

Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy)\cos t + (x^2 - 2y)e^t$$

$$= (2\sin t\, e^t)\cos t + (\sin^2 t - 2e^t)e^t$$

$$w = \sin^2 t\, e^t - e^{2t}$$

$$\frac{dw}{dt} =$$

Let $w = f(x,y)$, with $x = g(s,t)$ and $y = h(s,t)$:

$$\boxed{\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$

<u>Ex.</u> Let $w = x^2y^2$, with $x = s^2 + t^2$ and $y = \dfrac{s}{t}$.

Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

$$= 2xy^2 \cdot 2s + 2x^2y \cdot \frac{1}{t}$$

$$= 2(s^2+t^2)\left(\frac{s}{t}\right)^2 \cdot 2s + 2(s^2+t^2)^2 \cdot \frac{s}{t} \cdot \frac{1}{t}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= 2xy^2 \cdot 2t + 2x^2y\left(\frac{-s}{t^2}\right)$$

$$= 2(s^2+t^2)\left(\frac{s}{t}\right)^2 \cdot 2t + 2(s^2+t^2)^2 \cdot \frac{s}{t}\left(\frac{-s}{t^2}\right)$$

Ex. Let $\overbrace{y^3 + y^2 - 5y - x^2 + 4}^{F} = 0$, find $\dfrac{dy}{dx}$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2+2y-5} = \frac{-F_x}{F_y} \qquad F(x,y) = y^3 + y^2 - 5y - x^2 + 4$$

<u>Thm.</u> If $F(x,y) = 0$ defines y implicitly as a function of x, then

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

<u>Ex.</u> Find y' if $x^3 + y^3 = 6xy$.

$$\underbrace{x^3 + y^3 - 6xy}_{F} = 0$$

$$y' = \frac{-F_x}{F_y} = \frac{-(3x^2 - 6y)}{3y^2 - 6x}$$

Thm. If $F(x,y,z) = 0$ defines z implicitly as a function of x and y, then

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

<u>Ex.</u> If $\overbrace{x^3 + y^3 + z^3 + 6xyz}^{F} = 1$, find the first partial derivatives of z.

$$z_x = \frac{-F_x}{F_z} = \frac{-(3x^2+6yz)}{3z^2+6xy}$$

$$z_y = \frac{-F_y}{F_z} = \frac{-(3y^2+6xz)}{3z^2+6xy}$$

Directional Derivatives and Gradients

The gradient of a function $f(x,y)$ is the vector:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

→ This is sometimes written **grad** f

→ Note that ∇f is in 2-D, even though $f(x,y)$ is in 3-D

Ex. Let $f(x,y) = \sin x + e^{xy}$, find $\nabla f(0,1)$.

$$\nabla f = \langle \cos x + e^{xy} \cdot y \,,\, e^{xy} \cdot x \rangle$$

$$\nabla f(0,1) = \langle \cos 0 + e^{0\cdot 1} \cdot 1 \,,\, e^{0 \cdot 1} \cdot 0 \rangle$$

$$= \langle 2 \,,\, 0 \rangle$$

The rate of change in the x-direction (toward the vector **i**) is f_x

The rate of change in the y-direction (toward the vector **j**) is f_y

What if we want the rate of change in the direction of an arbitrary vector **u**?

→ This is called the directional derivative in the direction of **u**.

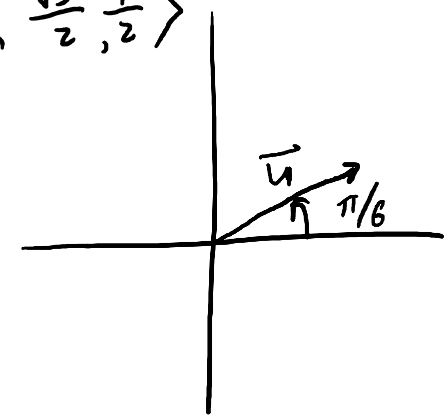
Thm. The directional derivative of f in the direction of unit vector **u** is

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

Ex. Find the $D_{\mathbf{u}}f$ if $f(x,y) = x^3 - 3xy + 4y^2$ and if $\mathbf{u}$ is the vector in the direction of $\theta = \frac{\pi}{6}$. What is $D_{\mathbf{u}}f(1,2)$?

$$\vec{u} = \langle \cos\tfrac{\pi}{6}, \sin\tfrac{\pi}{6} \rangle = \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$

$$\nabla f = \langle 3x^2 - 3y, -3x + 8y \rangle$$



$$D_u f = \langle 3x^2 - 3y, -3x + 8y \rangle \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$
$$= \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y)$$

$$D_u f(1,2) = \frac{\sqrt{3}}{2}(3 \cdot 1^2 - 3 \cdot 2) + \frac{1}{2}(-3 \cdot 1 + 8 \cdot 2)$$
$$= \frac{\sqrt{3}}{2}(-3) + \frac{1}{2}(13)$$
$$= \frac{-3\sqrt{3}}{2} + \frac{13}{2}$$

Ex. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at the point (2,-1) in the direction from A(2,-1) to B(4,4).

$\vec{AB} = \langle 2, 5 \rangle$

$\nabla f = \langle 2xy^3, 3x^2y^2 - 4 \rangle$

$\|\vec{AB}\| = \sqrt{4 + 25} = \sqrt{29}$

$\nabla f(2,-1) = \langle 2 \cdot 2(-1)^3, 3 \cdot 2^2(-1)^2 - 4 \rangle$

$= \langle -4, 8 \rangle$

$\vec{u} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

$$D_u f = \langle -4, 8 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle = \frac{-8}{\sqrt{29}} + \frac{40}{\sqrt{29}} = \boxed{\frac{32}{\sqrt{29}}}$$

i. If $\nabla f = \mathbf{0}$, then $D_{\mathbf{u}} f(x,y) = 0$ for all $\mathbf{u}$.

ii. The direction of maximum increase of f is given by ∇f, and the maximum value of $D_{\mathbf{u}} f$ is $|\nabla f|$.

iii. The direction of minimum increase of f is given by $-\nabla f$, and the minimum value of $D_{\mathbf{u}} f$ is $-|\nabla f|$.

Ex. If $f(x,y) = xe^y$, find the direction in which f has the maximum rate of change. What is this maximum rate?

$$\nabla f = \langle e^y, xe^y \rangle$$

$$\|\nabla f\| = \sqrt{e^{2y} + x^2 e^{2y}}$$

Ex. The temperature at a point on a plane is given by the equation $T(x,y) = 20 - 4x^2 - y^2$. In what direction from (2,-3) does the temperature increase most rapidly?

$$\nabla T = \langle -8x, -2y \rangle$$

$$\nabla T(2,-3) = \langle -16, 6 \rangle$$

Ex. A heat-seeking particle is located at (2,-3) on a metal plate whose temperature is given by $T(x,y) = 20 - 4x^2 - y^2$. Find the path of the particle as it moves in the direction of maximum temperature increase.

$$\nabla T = \langle \underset{\Delta x}{-8x}, \underset{\Delta y}{-2y} \rangle$$

$$\frac{dy}{dx} = \frac{-2y}{-8x}$$

$$\frac{dy}{dx} = \frac{y}{4x}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{4x}\,dx$$

$$e^{\ln|y|} = e^{\frac{1}{4}\ln|x| + C}$$

$$|y| = e^{\frac{1}{4}\ln|x|} \cdot e^{C}$$

$$y = D\sqrt[4]{|x|}$$

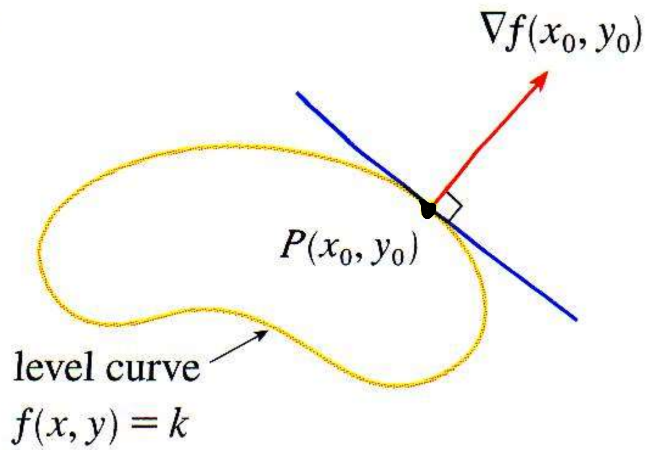
$$-3 = D\sqrt[4]{2}$$

$$D = \frac{-3}{\sqrt[4]{2}}$$

$$\boxed{y = \frac{-3}{\sqrt[4]{2}}\sqrt[4]{|x|}}$$

This works because the gradient always points toward the nearest peak or away from the nearest valley…

Thm. If f is differentiable at (x_0,y_0) and $\nabla f(x_0,y_0) \neq 0$, then $\nabla f(x_0,y_0)$ is normal to the level curve through (x_0,y_0).



Ex. Find a normal vector to the level curve corresponding to $c = 36$ of $f(x,y) = 5x^2 + y^2$ at (2,4).

$$\nabla f = \langle 10x, 2y \rangle$$

$$\nabla f(2,4) = \langle 20, 8 \rangle$$