Change of Variables

In 2-D, $\int_{1}^{3} x\sqrt{x^2-1}\,dx = \int_{0}^{8} \sqrt{u}\left(\frac{1}{2}\right)du$

→ We changed the variables to make the integral easier.

For double integrals, we change the variables to u and v.

→ Just like in 2-D where $dx = ?? \, du$, we need to have $dy \, dx = ?? \, du \, dv$.

→ This extra stuff is called the <u>Jacobian</u>

Def. If $x = g(u,v)$ and $y = h(u,v)$, then the Jacobian of x and y with respect to u and v is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(absolute value of the determinant)

<u>Ex.</u> Find the Jacobian for the change of variables $x = \overset{r\cos\theta}{u \cos v}$ and $y = \overset{r\sin\theta}{u \sin v}$.

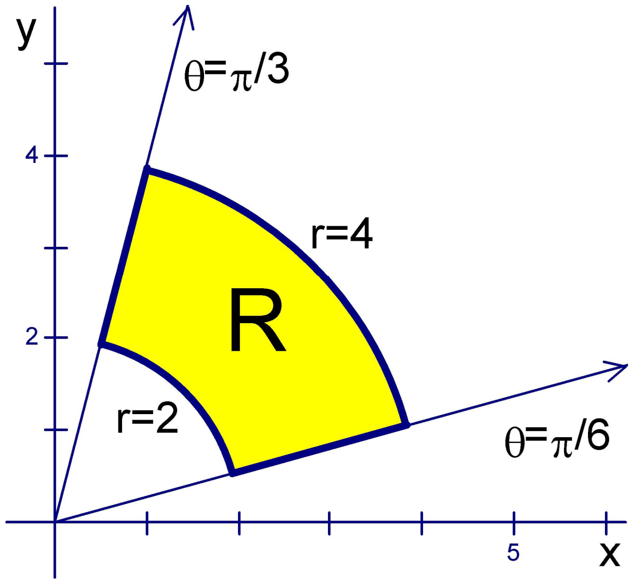
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix}$$

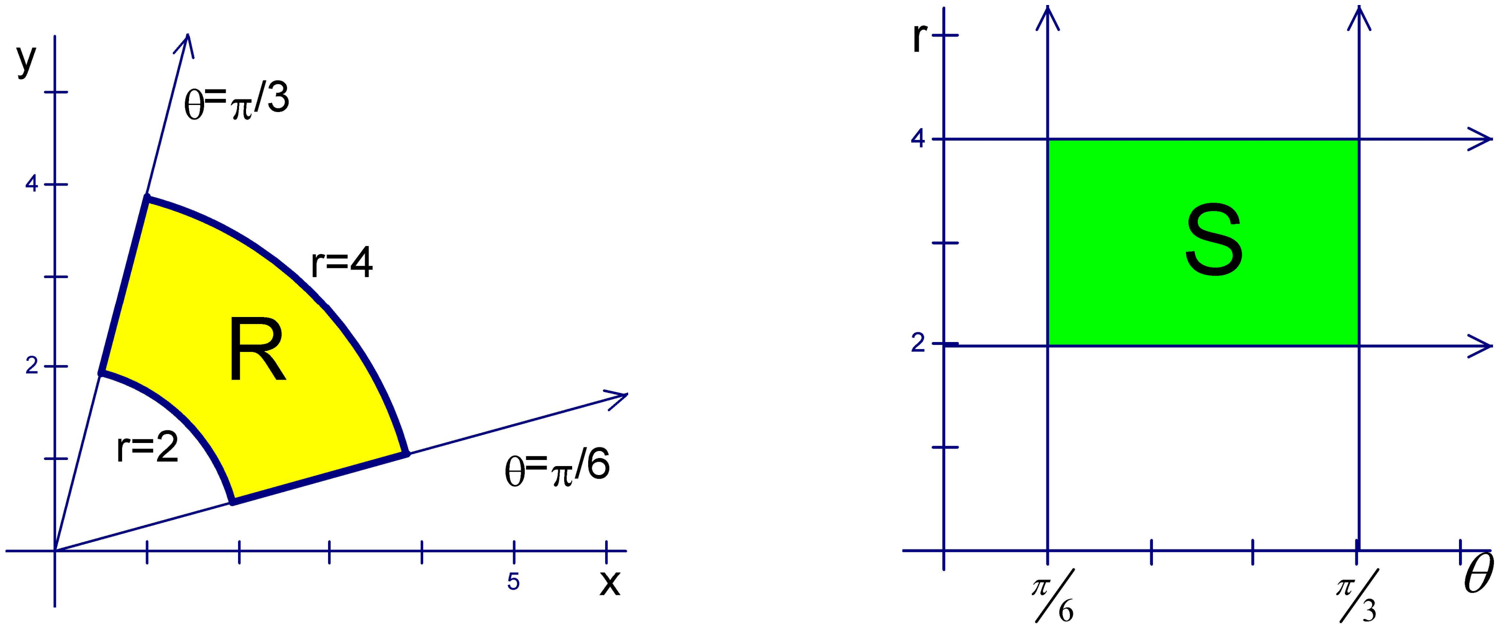
$$= u\cos^2 v + u\sin^2 v = \underset{r}{u}$$

This means that

$$\iint\limits_R f(x,y)\,dy\,dx = \iint\limits_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

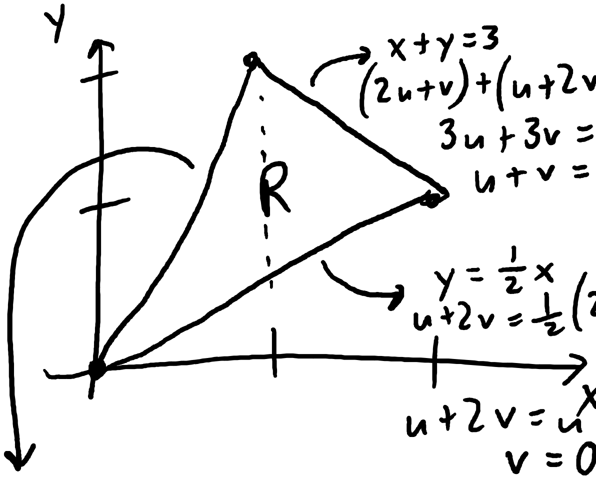
where R is in the xy-plane and S is in the $r\theta$-plane.


y
θ=π/3
4
r=4
R
2
r=2
θ=π/6
5
x



The point is to make a simpler, more easily integrated region.

<u>Ex.</u> Use the change of variables $x = 2u + v$, $y = u + 2v$ to evaluate the integral $\iint\limits_R y\,dA$ where R is the triangular region with vertices (0,0), (2,1), and (1,2).



$x + y = 3$

$(2u+v) + (u+2v) = 3$

$3u + 3v = 3$

$u + v = 1$

$y = \frac{1}{2}x$

$u + 2v = \frac{1}{2}(2u+v)$

$u + 2v = u + \frac{1}{2}v$

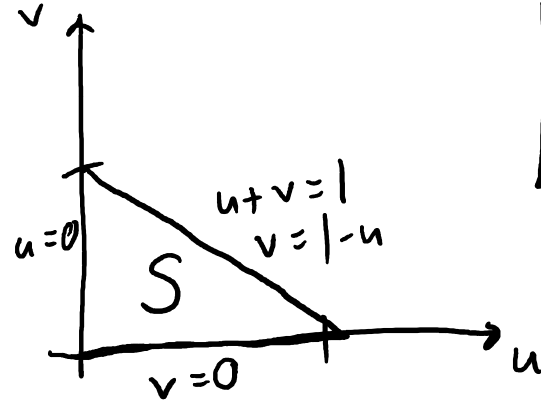
$v = 0$

$y = 2x$

$u + 2v = 2(2u+v)$

$u + 2v = 4u + 2v$

$u = 0$

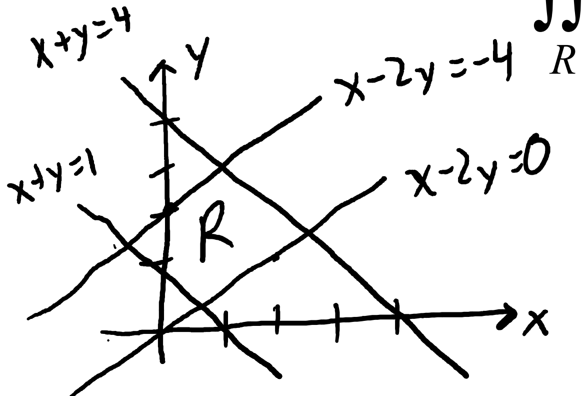


$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3}$$

$$\iint\limits_R y\,dA = \iint\limits_S (u+2v)\,3\,dv\,du = \int_0^1 \int_0^{1-u} (u+2v)\,3\,dv\,du$$

$\vdots$

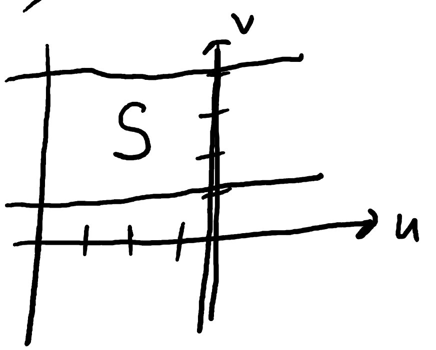
Ex. Let R be the region bounded by $x - 2y = 0$, $x - 2y = -4$, $x + y = 1$, and $x + y = 4$. Find a change of variables, sketch the image in the uv-plane, calculate the Jacobian, and evaluate $\iint_R 3xy\,dA$.



$$\begin{aligned} u &= x - 2y \\ - \quad v &= x + y \\ \hline u - v &= -3y \\ y &= \tfrac{1}{3}(v-u) \end{aligned}$$

$$\begin{aligned} u &= x - 2y \\ \xrightarrow{\times 2} \quad 2v &= 2x + 2y \\ \hline u + 2v &= 3x \\ x &= \tfrac{1}{3}(u+2v) \end{aligned}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$



$$\iint_R 3xy\,dA = \int_{-4}^{0} \int_{1}^{4} 3 \cdot \tfrac{1}{3}(u+2v) \cdot \tfrac{1}{3}(v-u)\,\tfrac{1}{3}\,dv\,du$$

Def. If $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = j(u,v,w)$, then the Jacobian of x, y, and z with respect to u, v and w is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$