Vector Fields

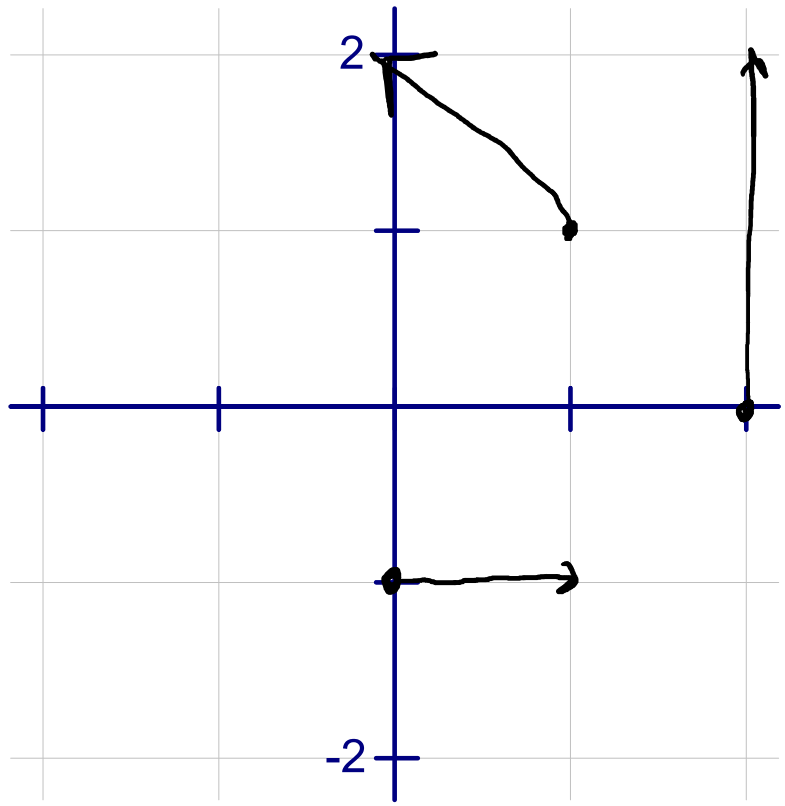
Def. A vector field is a function **F** that assigns to every point in the *xy*-plane a two-dimensional vector $\mathbf{F}(x,y)$.

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

→These are the same as vector-valued functions, but now they are multi-variable.

→We can also have vector fields with three components, graphed in 3-space.

Ex. Sketch the vector field $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$.



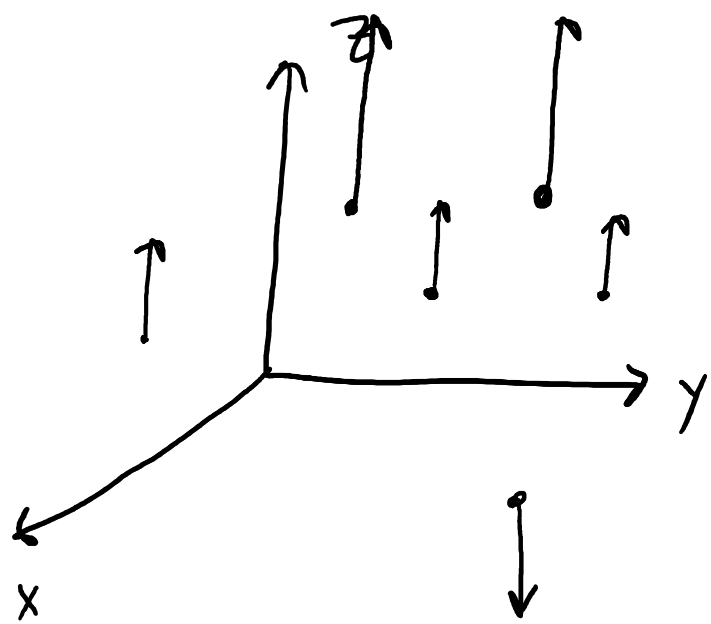
$\vec{F}(1,1) = \langle -1, 1 \rangle$

$\vec{F}(2,0) = \langle 0, 2 \rangle$

$\vec{F}(0,-1) = \langle 1, 0 \rangle$

<u>Ex.</u> Sketch the vector field $\mathbf{F}(x,y,z) = z\mathbf{k}$.

$= \langle 0, 0, z \rangle$



Vector fields can be used in some areas of physics:

- As fluid flows through a pipe, the velocity vector **V** can be found at any location within the pipe.
- The gravity created by a mass creates a vector field where every vector points at the mass, and the magnitudes get smaller as points are chosen further from the mass.

We've already worked with an example of a vector field…

GRADIENT

<u>Ex.</u> Let $f(x,y) = 16x^2y^3$, find ∇f.

$$\nabla f = \underbrace{\langle 32xy^3,\ 48x^2y^2 \rangle} = \vec{F}(x,y)$$

Def. A vector field **F** is called conservative if there is some function f such that $\mathbf{F} = \nabla f$. The function f is called the potential function of **F**.

Ex. Show that $\mathbf{F}(x,y) = 2x\mathbf{i} + y\mathbf{j}$ is conservative.

$$\nabla f = \langle \underset{f_x}{\underset{\uparrow}{2x}}, \underset{f_y}{\underset{\uparrow}{y}} \rangle$$

$f_x = 2x$

$f = x^2 + C_x$

$f_y = y$

$f = \frac{1}{2}y^2 + C_y$

$$f(x,y) = x^2 + \frac{1}{2}y^2$$

Thm. The vector field

$$\mathbf{F}(x,y) = \overset{f_x}{P(x,y)}\,\mathbf{i} + \overset{f_y}{Q(x,y)}\,\mathbf{j}$$

is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$f_{xy} = f_{yx}$$

Ex. Determine if the vector field is conservative.

a) $\mathbf{F} = x^2y\mathbf{i} + xy\mathbf{j}$

$P_y = x^2$ $\quad\therefore$ not conserv.

$Q_x = y$

b) $\mathbf{F} = 2x\mathbf{i} + y\mathbf{j}$

$P_y = 0$ $\quad\therefore$ conserv.

$Q_x = 0$

Ex. Show that $\mathbf{F}(x,y) = \underline{2xy^3}\mathbf{i} + (\underline{3x^2y^2 + 2y^3})\mathbf{j}$ is conservative, then find its potential function.

$P_y = 6xy^2$ $\therefore$ conserv.

$Q_x = 6xy^2$

$f_x = 2xy^3$

$f = x^2y^3$

$f_y = 3x^2y^2 + 2y^3$

$f = x^2y^3 + \frac{1}{2}y^4$

$$f(x,y) = x^2y^3 + \frac{1}{2}y^4$$

Curl and Divergence

<u>Def.</u> The <u>curl</u> of $\mathbf{F}(x,y,z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F}$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

<u>Ex.</u> If $\mathbf{F} = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find curl $\mathbf{F}$.

$$\vec{F} = \langle xz, xyz, -y^2 \rangle$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -y^2 \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xz & -y^2 \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz & xyz \end{vmatrix}$$

$$= \hat{i}(-2y - xy) - \hat{j}(0 - x) + \hat{k}(yz - 0)$$

$$= \langle -2y - xy, x, yz \rangle$$

Thm. A vector field $\mathbf{F}$ is conservative if and only if curl $\mathbf{F} = \mathbf{0}$.

So the last vector field was not conservative.

If curl $\mathbf{F} = \mathbf{0}$ at a point, then we say that the vector field is irrotational at the point.

Ex. Show that $\vec{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ is conservative, and find its potential function.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \hat{i}(6xyz^2 - 6xyz^2) - \hat{j}(3y^2z^2 - 3y^2z^2) + \hat{k}(2yz^3 - 2yz^3) = \vec{0}$$

✓

$f_x = y^2z^3$ | $f_y = 2xyz^3$ | $f_z = 3xy^2z^2$

$f = xy^2z^3$ | $f = xy^2z^3$ | $f = xy^2z^3$

$$\boxed{f(x,y,z) = xy^2z^3}$$

Def. The divergence of $\mathbf{F}(x,y,z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$$

Ex. If $\mathbf{F} = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find div $\mathbf{F}$.

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle xz, xyz, -y^2 \right\rangle$$

$$= z + xz + 0$$

Note: curl $\mathbf{F}$ is a vector

div $\mathbf{F}$ is a scalar

Thm. div $(\underbrace{\text{curl } \mathbf{F}}_{\text{vector}}) = 0$

$$\text{curl}(\underbrace{\text{div } \vec{F}}_{\text{scalar}}) = ?$$

Ex. Show that $\mathbf{F} = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ can't be written as the curl of another vector field, that is $\mathbf{F} \neq \text{curl } \mathbf{G}$.

$$\text{div } \vec{F} \neq \text{div}(\text{curl } \vec{G})$$

$$x + xz \neq 0$$

$$\text{div } \vec{F} = \text{div } \vec{H} \not\Rightarrow \vec{F} = \vec{H}$$

$$\vec{F} = \vec{H} \Longrightarrow \text{div } \vec{F} = \text{div } \vec{H}$$