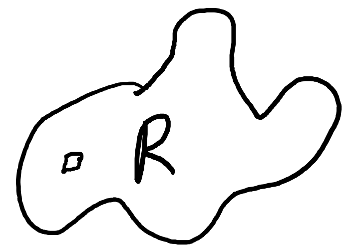
Line Integrals

$\int_a^b f(x)\,dx \rightarrow$ integral over an interval

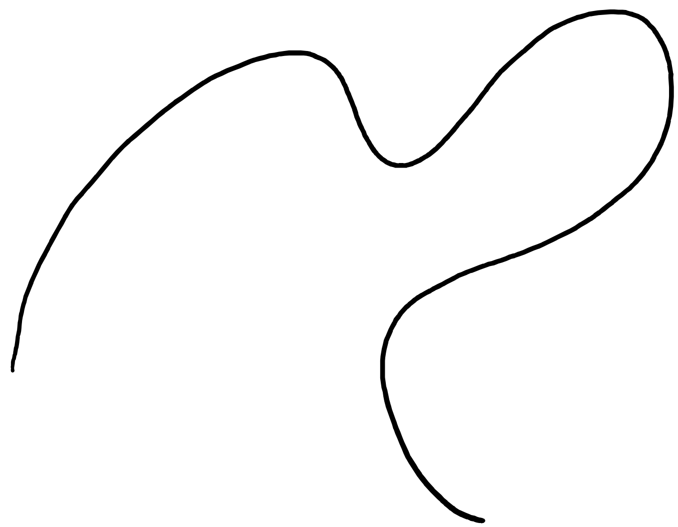


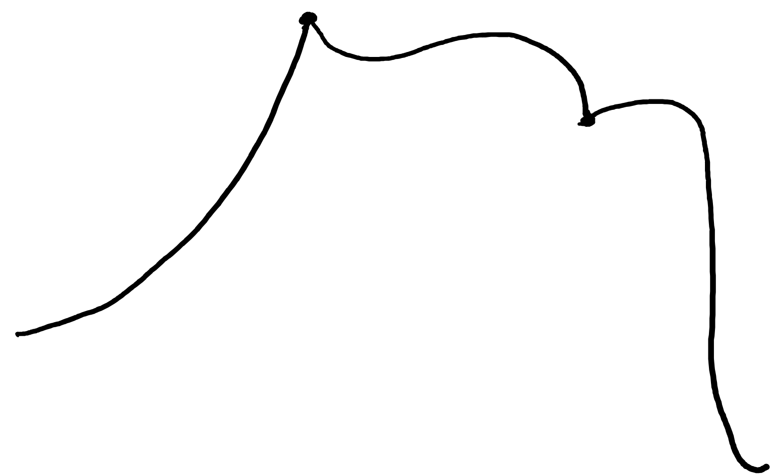
$\iint_R f(x,y)\,dA \rightarrow$ integral over a region

$\int_C f(x,y)\,ds \rightarrow$ line integral over the curve C

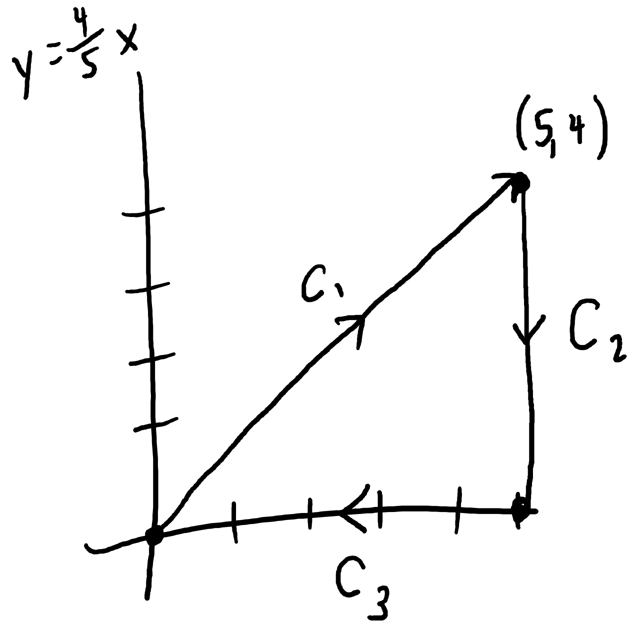
Remember that s is the arc length.

Def. A curve C is piecewise smooth on an interval if the interval can be broken into a finite number of pieces, on each of which C is smooth.





Ex. Find a piecewise smooth parameterization of the path from (0,0) to (5,4) to (5,0) to (0,0).



C_1: $\vec{r}_1(t) = \langle t, \frac{4}{5}t \rangle$

$0 \le t \le 5$

C_2: $\vec{r}_2(t) = \langle 5, -t \rangle$

$-4 \le t \le 0$

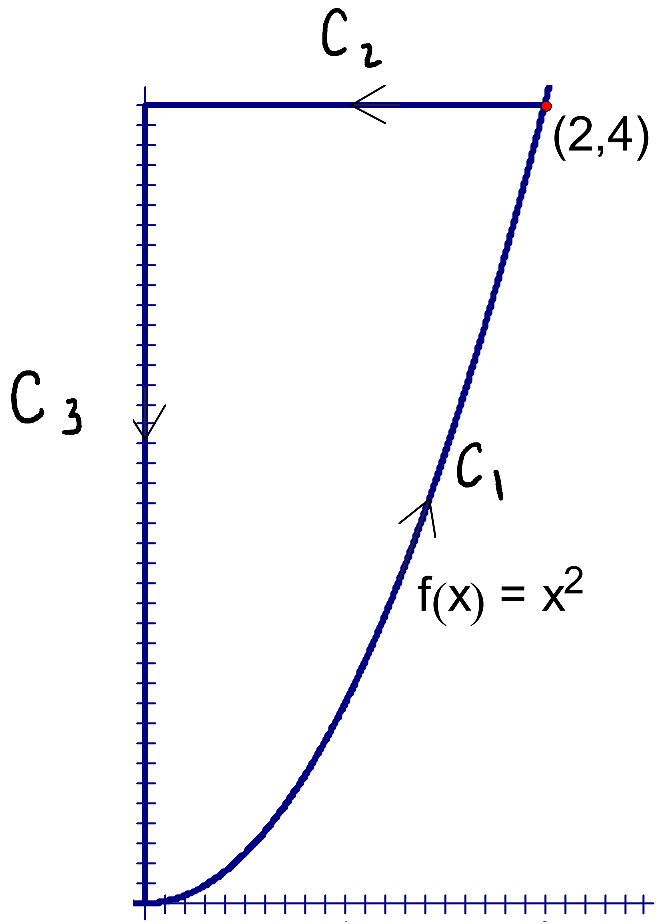
C_3: $\vec{r}_3(t) = \langle 5 - t, 0 \rangle$

$0 \le t \le 5$

$\langle e^t, \frac{4}{5}e^t \rangle$

$\langle 5t, 4t \rangle \quad 0 \le t \le 1$

Ex. Find a piecewise smooth parameterization of the path below.



$C_1: \vec{r}_1(t) = \langle t, t^2 \rangle$

$0 \le t \le 2$

$C_2: \vec{r}_2(t) = \langle -t, 4 \rangle$

$-2 \le t \le 0$

$C_3: \vec{r}_3(t) = \langle 0, -t \rangle$

$-4 \le t \le 0$

Thm. Let f be a continuous function containing a smooth curve C. If C can be written $\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \le t \le b$, then

$$\int_C f(x,y)\,ds = \int_a^b f(x,y)\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

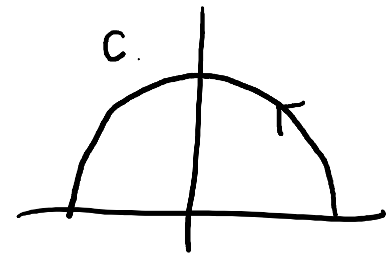
→ Can be extended to 3-D

→ Note if $f(x,y) = 1$, $\int_C 1\,ds =$ length of C

Ex. Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle, oriented ccw.

$$\int_C (2+x^2y)\,ds = \int_0^\pi (2+\cos^2 t\sin t)\sqrt{(-\sin t)^2+(\cos t)^2}\,dt$$

$\vdots$



$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le \pi$

Ex. Evaluate $\int_C (2+x)\,ds$, where C is represented by

$$\vec{r}(t) = \left\langle \underline{t}, \underline{\tfrac{4}{3}t^{3/2}}, \underline{\tfrac{1}{2}t^2} \right\rangle, 0 \le t \le 2$$

$$\int_C (2+x)\,ds = \int_0^2 (2+t)\sqrt{1^2 + \left(2t^{1/2}\right)^2 + t^2}\,dt$$

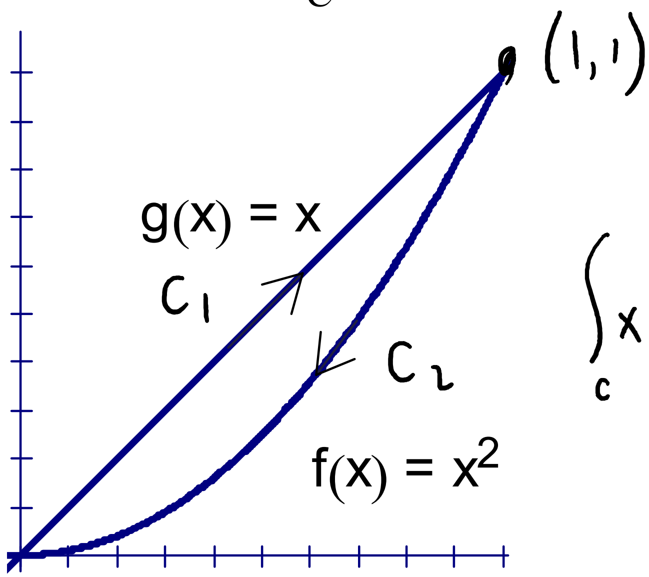
$$= \int_0^2 (2+t)\sqrt{t^2+4t+1}\,dt = \int_*^{**} u^{1/2}\tfrac{1}{2}\,du = \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big|_*^{**}$$

$$\boxed{\begin{aligned} u &= t^2+4t+1 \\ du &= (2t+4)\,dt \\ \tfrac{1}{2}du &= (2+t)\,dt \end{aligned}}$$

$$= \tfrac{1}{3}\left(t^2+4t+1\right)^{3/2}\Big|_0^2$$

$$= \tfrac{1}{3}\left(13^{3/2}-1\right)$$

Ex. Set up $\int_C x\,ds$, where C is shown below.



$$\int_C x\,ds = \int_0^1 t\sqrt{1^2+1^2}\,dt + \int_{-1}^0 -t\sqrt{(-1)^2+(2t)^2}\,dt$$

C_1: $\vec{r}_1 = \langle t, t \rangle$
$0 \le t \le 1$

C_2: $\vec{r}_2 = \langle -t, t^2 \rangle$
$-1 \le t \le 0$

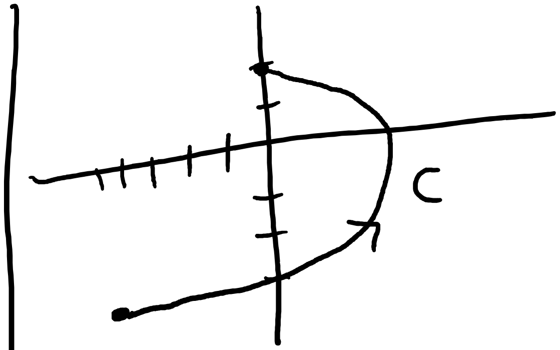
<u>Thm.</u> Let **F** be a continuous vector field containing a smooth curve C. If C can be written $\vec{r}(t)=\langle x(t),y(t)\rangle$ for $a \le t \le b$, then

$$\int_C \vec{F}\cdot d\vec{r}=\int_a^b \vec{F}\left(\vec{r}(t)\right)\cdot\vec{r}\,'(t)\,dt$$

Ex. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle y^2, x \rangle$ and C is the portion of the parabola $x = 4 - y^2$ from (-5,-3) to (0,2).

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-3}^{2} \underbrace{\langle t^2, 4-t^2 \rangle}_{\vec{F}} \cdot \underbrace{\langle -2t, 1 \rangle}_{d\vec{r}} \, dt$$

$$= \int_{-3}^{2} \left(-2t^3 + 4 - t^2\right) dt$$



$\vec{F} = \langle 4-t^2, t \rangle$

$-3 \le t \le 2$

In the last example, we would have gotten a different answer if we integrated over the segment that connects the endpoints.

→ The line integral depends on path, not just endpoints.

Please note that you will get the same answer regardless of parameterization, but changing the direction will give you the opposite answer.

<u>Ex.</u> Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \frac{-1}{2}x\hat{i} - \frac{1}{2}y\,\hat{j} + \frac{1}{4}\hat{k}$ and C is parameterized by $\vec{r}(t) = \langle \cos t, \sin t, t\rangle, 0 \le t \le 3\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{3\pi} \left\langle -\tfrac{1}{2}\cos t, -\tfrac{1}{2}\sin t, \tfrac{1}{4}\right\rangle \cdot \langle -\sin t, \cos t, 1\rangle\, dt$$

$$= \int_0^{3\pi} \tfrac{1}{2}\sin t\cos t - \tfrac{1}{2}\sin t\cos t + \tfrac{1}{4}\, dt = \int_0^{3\pi} \tfrac{1}{4}\, dt$$

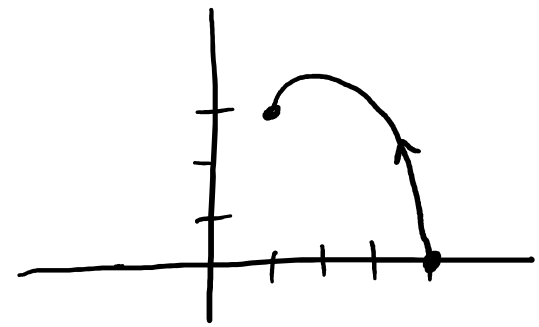
$$= \tfrac{1}{4}t\Big|_0^{3\pi} = \boxed{\frac{3\pi}{4}}$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then

$$\int_C \overrightarrow{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

Ex. Evaluate $\int_C y\,dx + x^2\,dy$, where C is the portion of the curve $y = 4x - x^2$ from (4,0) to (1,3).

$\vec{F} = \langle y, x^2 \rangle$



$\vec{r} = \langle -t, -4t - t^2 \rangle$

$-4 \le t \le -1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-4}^{-1} \langle -4t - t^2, t^2 \rangle \cdot \langle -1, -4 - 2t \rangle \, dt$$

$$= \int_{-4}^{-1} 4t + t^2 - 4t^2 - 2t^3 \, dt$$

$$= \int_{-4}^{-1} -2t^3 - 3t^2 + 4t \, dt = -\tfrac{1}{2}t^4 - t^3 + 2t^2 \Big|_{-4}^{-1}$$

$$= \left(-\tfrac{1}{2} + 1 + 2\right) - \left(-128 + 64 + 32\right) = 2.5 + 32 = 34.5$$

Ex. Find the work done by the field $\mathbf{F} = y\mathbf{i} + x^2\mathbf{j}$ on a particle moving along the curve $y = 4x - x^2$ from (4,0) to (1,3).

$$\text{Work} = \int \text{Force} = \int_C \vec{F} \cdot d\vec{r} = 34.5$$