Math 206 – Differential Equations

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Student Learning Outcomes

- Successful students will be able to compare first- and second-order differential equations, solve these equations using appropriate techniques including constructing solutions using series and matrices, and apply them to problems in science and engineering.

Definitions and Terms

A <u>differential equation</u> (diff. eq., DE) is an equation that involves x, y, and some derivatives of y.

y''

$$\frac{dy}{dx} + 5y = e^x \qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$

$$x + y''' = 2xy' + y$$

These are called <u>ordinary differential equations</u> (ODEs) because y is a function of only x.

Equations using partial derivatives are called partial differential equations (PDEs).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial u}{\partial y} + \sin x \frac{\partial u}{\partial x} = 0$$

We will only be studying ODEs in this course.

The <u>order</u> of a DE is the highest derivative in the equation.

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^5 = 0 \quad \leftarrow \text{Order 3}$$

y''''' $\qquad$ $y^{(5)}$

$$\frac{d^3p}{dt^3} + \left(\frac{dp}{dt}\right)^5 = 0$$

A DE is <u>linear</u> if it is linear in $y, y', y'', \ldots, y^{(n)}$

i.e. Each term has a coefficient that is a function of only x.

<u>Linear</u>

$$y'' - 2y' + y = 0$$

$$\frac{d^3y}{dx^3} + x^2\frac{dy}{dx} - 5y = e^x$$

$$\frac{(y-x)\cancel{dx}}{\cancel{dx}} + \frac{4x\,dy}{dx} = 0$$

$$y - x + 4x\frac{dy}{dx} = 0$$

<u>Non-linear</u>

$$(y-1)\,y' + 2y = e^x$$

$$\frac{d^2y}{dx^2} + \sin(y) = 0$$

$$\frac{d^4y}{dx^4} + y^2 = 0$$

Practice Problems

State the order of the DE and determine if it is linear or non-linear.

1) $x\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^4 + y = 0$

2) $\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + u = \cos r$

3) $\dfrac{d^2R}{dt^2} = -\dfrac{k}{R}$

$f(x)y + f_1(x)y' + f_2(x)y'' + \ldots = g(x)$

$y(x)$, defined on an interval I, is a <u>solution</u> of a DE if, when substituted into the DE, it reduces the equation to an identity.

<u>Ex.</u> Verify that the given function is a solution to the given DE.

$$y' = xy^{1/2}, \quad y = \frac{1}{16}x^4 \qquad y' = \frac{1}{4}x^3$$

$$\frac{1}{4}x^3 = x\left(\frac{1}{16}x^4\right)^{1/2}$$

$$\frac{1}{4}x^3 = x \cdot \frac{1}{4}x^2$$

✓

Ex. Verify that the given function is a solution to the given DE.

$$y'' - 2y' + y = 0, \; y = xe^x \longrightarrow$$

$y' = xe^x + e^x \cdot 1$

$= e^x(x+1)$

$y'' = e^x \cdot 1 + (x+1)e^x$

$= e^x(x+2)$

$e^x(x+2) - 2e^x(x+1) + xe^x = 0$

$e^x(x+2-2x-2+x) = 0$

$e^x \cdot 0 = 0$

✓

Notice that $y = 0$ is a solution to both DEs. This is called the trivial solution.

Ex. Find values for m that would make $y = e^{mx}$ a solution of the DE $2y'' + 7y' - 4y = 0$.

$y' = me^{mx}$

$y'' = m^2e^{mx}$

$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = 0$

$e^{mx}(2m^2 + 7m - 4) = 0$

$e^{mx}(2m-1)(m+4) = 0$

$e^{mx} = 0$ $\quad$ $2m - 1 = 0$ $\quad$ $m + 4 = 0$

$\times$ $\quad$ $m = \frac{1}{2}$ $\quad$ $m = -4$

Practice Problems

Verify that $y = \tfrac{6}{5} - \tfrac{6}{5}e^{-20t}$ is a solution to the DE $\tfrac{dy}{dt} + 20y = 24$

The interval on which a solution is defined is called the <u>interval of definition</u>.

a.k.a. interval of existence

a.k.a. interval of validity

a.k.a. domain of the solution

Consider $y = \frac{1}{x}$, which is a solution to the DE $xy' + y = 0$.

As a function, $y = \frac{1}{x}$ has domain $(-\infty, 0) \cup (0, \infty)$

A solution must be defined on an interval, so we must choose the interval of definition, either $(-\infty, 0)$ or $(0, \infty)$ or some subinterval.

Which one we choose depends on other info that we could be given.

Ex. Given $x^2 + y^2 = 25$, find $\frac{dy}{dx}$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

So $x^2 + y^2 = 25$ is a solution to the differential equation $\frac{dy}{dx} = \frac{-x}{y}$. This is called an <u>implicit solution</u>.

The <u>explicit solution</u> could be $y = \sqrt{25 - x^2}$ or $y = -\sqrt{25 - x^2}$

The one we choose depends on other information that we may be given.

Not all implicit solutions can be written explicitly.

Note that $x^2 + y^2 = c$ would also be a solution for the DE $\frac{dy}{dx} = \frac{-x}{y}$ for any $c \geq 0$.

$x^2 + y^2 = c$ is called a <u>one-parameter family of solutions</u>

When solving an n^{th} order DE, we will want to find an n-parameter family of solutions.

Ex. Show that $x = c_1 \sin 4t$ is a solution to the linear DE $x'' + 16x = 0$.

$x' = 4c_1 \cos 4t$

$x'' = -16c_1 \sin 4t$

$-16c_1 \sin 4t + 16c_1 \sin 4t = 0$

$0 = 0$

✓

A solution where we’ve chosen a value for the parameter is called a particular solution. With the parameters, we call it the general solution.

→ $x^2 + y^2 = 25$ was a particular solution

→ $x^2 + y^2 = c$ was a general solution

<u>Ex.</u> Show that $y = c_1 e^x + c_2 x e^x$ is a family of solutions of the DE $\underbrace{y'' - 2y' + y = 0}$.

$y = xe^x$

$c_1 = 0$

$c_2 = 1$

$y = xe^x$

$y = 0 \rightarrow c_1 = 0$

$c_2 = 0$

Ex. We saw that $y = \frac{1}{16}x^4$ was a solution to

particular

$y' = xy^{1/2}$. A family of solutions is

$c = 0$

$y = \left(\frac{1}{4}x^2 + c\right)^2$.

Note that we get our particular solution by setting $c = 0$.

$y = 0 \rightarrow c = ?$

But we saw that $y = 0$ is also a solution, and its not a member of the family of solutions.

This extra solution is called a singular solution.

Note that $y = cx^4$ is a family of solutions of the DE $xy' - 4y = 0$.

$y = 0 \to C = 0$
particular

However, a singular solution could be

$$y = \begin{cases} -x^4 & x > 0 \\ x^4 & x \le 0 \end{cases}$$

When we find a family of solutions, how can we know if it describes all solutions or if there are singular solutions…

$$\begin{cases} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 5x + 3y \end{cases}$$

is called a <u>system of DEs</u>. The solution is $x = f(t), y = g(t)$.

Ex. Verify that $x = e^{-5t}$, $y = 2e^{-5t}$ is a solution to the system of DEs

$$\begin{cases} \dfrac{dx}{dt} = 3x - 4y \\ \dfrac{dy}{dt} = 4x - 7y \end{cases}$$

$$-5e^{-5t} = 3e^{-5t} - 4(2e^{-5t})$$

$$-5e^{-5t} = e^{-5t}(3-8)$$

✓

$$-10e^{-5t} = 4e^{-5t} - 7(2e^{-5t})$$

$$-10e^{-5t} = e^{-5t}(4-14)$$

✓