Math 206 – Differential Equations Andy Rosen

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Student Learning Outcomes

• Successful students will be able to compare first- and second-order differential equations, solve these equations using appropriate techniques including constructing solutions using series and matrices, and apply them to problems in science and engineering.

Definitions and Terms

A <u>differential equation</u> (diff. eq., DE) is an equation that involves x, y, and some derivatives of y. $\frac{dy}{dx} + 5y = e^{x} \qquad \left(\frac{d^{2}y}{dx^{2}}\right) - \frac{dy}{dx} + 6y = 0$

$$\frac{dx^2}{dx} \frac{dx}{dx} = 2xy' + y$$

These are called <u>ordinary differential</u> <u>equations</u> (ODEs) because y is a function of only x. Equations using partial derivatives are called <u>partial differential equations</u> (PDEs).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \qquad \frac{\partial u}{\partial y} + \sin x \frac{\partial u}{\partial x} = 0$$

We will only be studying ODEs in this course.

The <u>order</u> of a DE is the highest derivative in the equation.

$$\frac{d^3 y}{dx^3} + \left(\frac{dy}{dx}\right)^5 = 0 \quad \leftarrow \text{Order 3}$$

y (5) y $\frac{d^{3} \rho}{dt^{3}} + \left(\frac{d\rho}{dt}\right)^{5} = 0$ A DE is <u>linear</u> if it is linear in $y, y', y'', ..., y^{(n)}$

i.e. Each term has a coefficient that is a function of only *x*.

Linear v'' - 2v' + y = 0 $\frac{d^3y}{dx^3} + x^2\frac{dy}{dx} - 5y = e^x$ $\underbrace{(\underline{v-x})dx}_{y-x} + \underbrace{4xdv}_{dx} = 0$ $\frac{\text{Non-linear}}{(y-1)y'+2y} = e^x$ $\frac{d^2y}{dx^2} + \sin(y) = 0$ $\frac{d^4y}{dx^4} + y^2 = 0$

Practice Problems

State the order of the DE and determine if it is linear or non-linear.

1)
$$x \frac{d^{3}y}{dx^{3}} - \left(\frac{dy}{dx}\right)^{4} + y = 0$$

2) $\frac{d^{2}u}{dr^{2}} + \frac{du}{dr} + u = \cos r$
3) $\frac{d^{2}R}{dt^{2}} = -\frac{k}{R}$ $f(x)y + f_{1}(x)y' + f_{2}(x)y'' + \dots = f(x)y''$

y(x), defined on an interval *I*, is a <u>solution</u> of a DE if, when substituted into the DE, it reduces the equation to an identity.

<u>Ex.</u> Verify that the given function is a solution to the given DE.

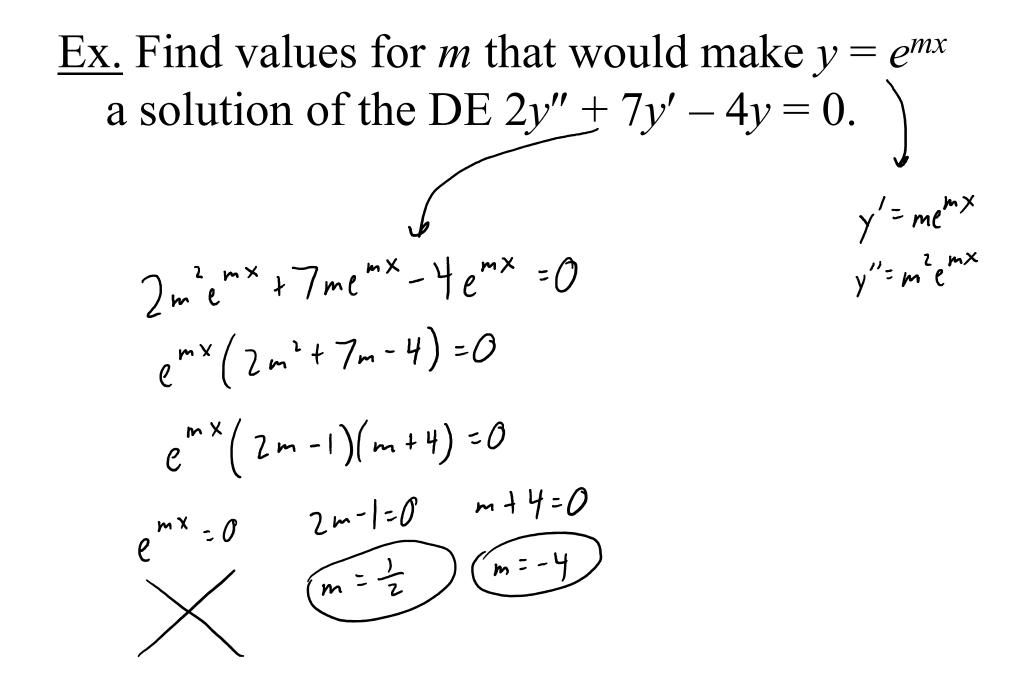
$$y' = xy^{\frac{1}{2}}, (y = \frac{1}{16}x^{4}) \qquad y' = \frac{1}{4}x^{3}$$

$$\frac{1}{4}x^{3} = x(\frac{1}{16}x^{4})^{\frac{1}{2}}$$

$$\frac{1}{4}x^{3} = x \cdot \frac{1}{4}x^{2}$$

<u>Ex.</u> Verify that the given function is a solution to the given DE.

Notice that y = 0 is a solution to both DEs. This is called the <u>trivial solution</u>.



Practice Problems

Verify that $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$ is a solution to the DE $\frac{dy}{dt} + 20y = 24$

The interval on which a solution is defined is called the <u>interval of definition</u>.

a.k.a. interval of existence

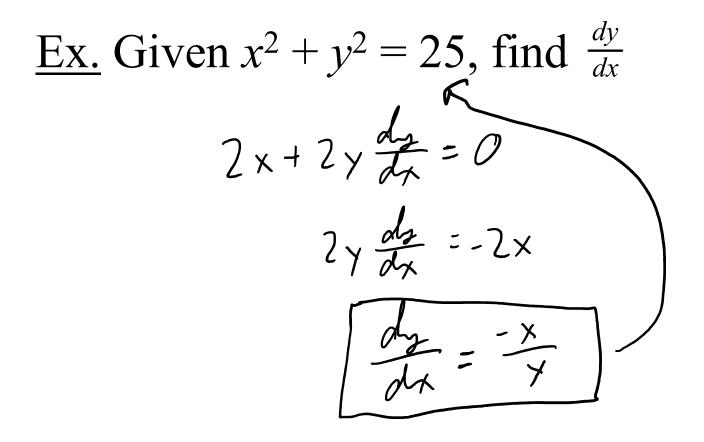
a.k.a. interval of validity

a.k.a. domain of the solution

Consider $y = \frac{1}{x}$, which is a solution to the DE xy' + y = 0.

As a function, $y = \frac{1}{x}$ has domain $(-\infty, 0) \cup (0, \infty)$

- A solution must be defined on an interval, so we must choose the interval of definition, either $(-\infty, 0)$ or $(0, \infty)$ or some subinterval.
- Which one we choose depends on other info that we could be given.



So $x^2 + y^2 = 25$ is a solution to the differential equation $\frac{dy}{dx} = \frac{-x}{y}$. This is called an <u>implicit</u> <u>solution</u>.

The <u>explicit solution</u> could be $y = \sqrt{25 - x^2}$ or $y = -\sqrt{25 - x^2}$

The one we choose depends on other information that we may be given.

Not all implicit solutions can be written explicitly.

Note that $x^2 + y^2 = c$ would also be a solution for the DE $\frac{dy}{dx} = \frac{-x}{y}$ for any $c \ge 0$.

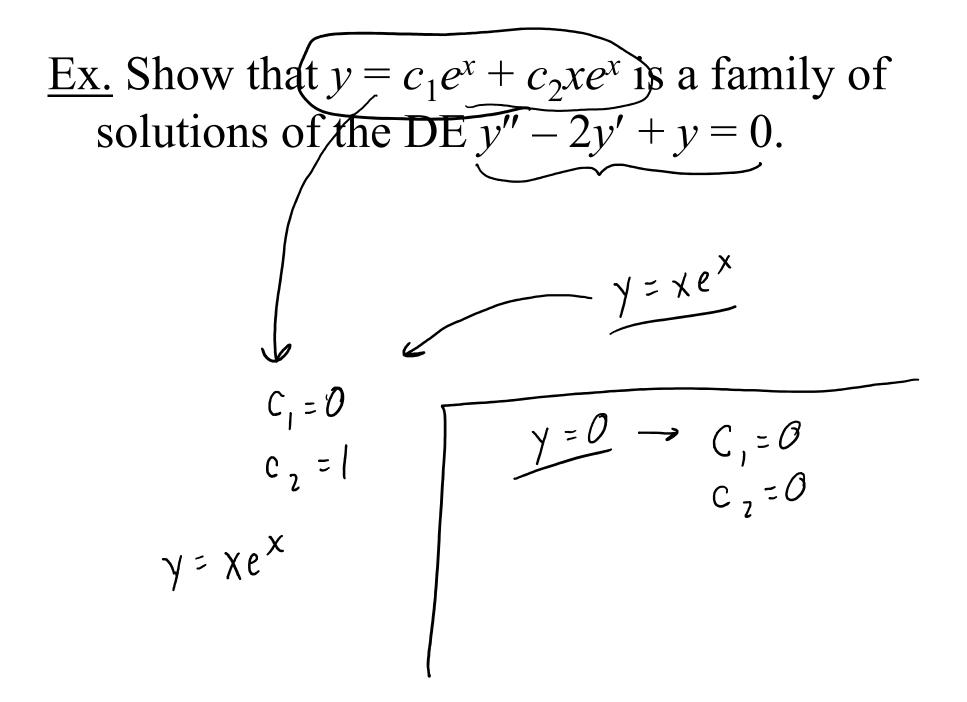
 $x^2 + y^2 = c$ is called a <u>one-parameter family of</u> <u>solutions</u>

When solving an *n*th order DE, we will want to find an *n*-parameter family of solutions.

<u>Ex.</u> Show that $x = c_1 \sin 4t$ is a solution to the linear DE x'' + 16x = 0. x'=4C, con4t x"=-16C, sin4t -16c, sin 4t + 16c, sin 4t = 00 = 0

A solution where we've chosen a value for the parameter is called a <u>particular solution</u>. With the parameters, we call it the <u>general</u> <u>solution</u>.

→
$$x^2 + y^2 = 25$$
 was a particular solution
→ $x^2 + y^2 = c$ was a general solution



Ex. We saw that
$$y = \frac{1}{16}x^4$$
 was a solution to
 $y' = xy^{\frac{1}{2}}$. A family of solutions is
 $y = \left(\frac{1}{4}x^2 + c\right)^2$.

Note that we get our particular solution by setting c = 0. But we saw that y = 0 is also a solution, and its

not a member of the family of solutions.

This extra solution is called a singular solution.

Note that $y \neq cx^4$ is a family of solutions of the DE xy' - 4y = 0. y = 0y = 0y = 0

However, a singular solution could be

$$y = \begin{cases} -x^4 & x > 0\\ x^4 & x \le 0 \end{cases}$$

When we find a family of solutions, how can we know if it describes all solutions or if there are singular solutions...

$$\begin{cases} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 5x + 3y \end{cases}$$

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is called a <u>system of DEs</u>. The solution is x = f(t), y = g(t).

<u>Ex.</u> Verify that $x = e^{-5t}$, $y = 2e^{-5t}$ is a solution to the system of DEs

