Warm-up Problems

1. Determine the order of the DE

$$\frac{d^3y}{dt^3} + t\frac{d^2y}{dt^2} + \left(\cos^2 t\right)y = t^3$$

   Is it linear?

2. Verify that $y = e^x + \frac{x}{3}$ is a solution to $y'''' - 4y''' + 3y = x$.

Initial Value Problems

A differential equation, together with values of the solution and its derivatives at a point x_0, is called an <u>initial value problem</u> (IVP).

<u>Ex.</u> $\frac{dy}{dx} = 7y,\ y(0) = 3$

An n^{th} order DE requires the value of y and its first $n - 1$ derivatives at x_0.

- It would have n parameters, so n pieces of information are needed.

Ex. $\frac{d^3y}{dx^3} - 6\frac{dy}{dx} + y = 0$, $y(0) = 3$, $y'(0) = 0$

$y''(0) = -1$

<u>Ex.</u> $y = ce^x$ is a family of solutions of $y' = y$. Find the particular solution that satisfies $y(1) = -2$.

$$-2 = Ce^1$$

$$C = \frac{-2}{e}$$

$$\boxed{y = \frac{-2}{e}e^x} = -2e^{x-1}$$

Ex. $y = \dfrac{1}{x^2 + c}$ is a family of solutions of $y' + 2xy^2 = 0$. Find the particular solution satisfying $y(0) = -1$.

$$-1 = \frac{1}{0 + c}$$

$$-1 = \frac{1}{c}$$

$$c = -1$$

$$\boxed{y = \frac{1}{x^2 - 1}}$$

$(-\infty, -1)$ $(-1, 1)$ $(1, \infty)$

Consider the interval of definition.

Ex. $x = c_1 \cos 4t + c_2 \sin 4t$ is a family of solutions for $x'' + 16x = 0$. Find the solution to the IVP

$$x'' + 16x = 0,\ x\left(\tfrac{\pi}{2}\right) = -2,\ x'\left(\tfrac{\pi}{2}\right) = 1$$

$\cos\left(4 \cdot \frac{\pi}{2}\right)$ $\sin\left(4 \cdot \frac{\pi}{2}\right)$

$-2 = c_1 \cos(2\pi) + c_2 \sin(2\pi)$

$-2 = c_1$

$\boxed{x = -2\cos 4t + \frac{1}{4}\sin 4t}$

$x' = -4c_1 \sin 4t + 4c_2 \cos 4t$

$1 = -4c_1 \sin 2\pi + 4c_2 \cos 2\pi$

$1 = 4c_2$

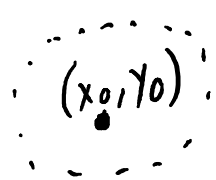
$c_2 = \frac{1}{4}$

We saw that $y = \frac{1}{16}x^4$ is a solution to the IVP $y' = xy^{1/2}$, $y(0) = 0$, but so is the solution $y = 0$.

- Can there be two different solutions to the same IVP?

Thm. Existance of a Unique Solution (1st Order DE)

If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in some open region containing (x_0,y_0), then there is an interval I in the region and a unique function $y(x)$ defined on I that is a solution to the IVP $\underbrace{\frac{dy}{dx} = f(x,y)}$, $\underbrace{y(x_0) = y_0}_{(x_0,\, y_0)}$

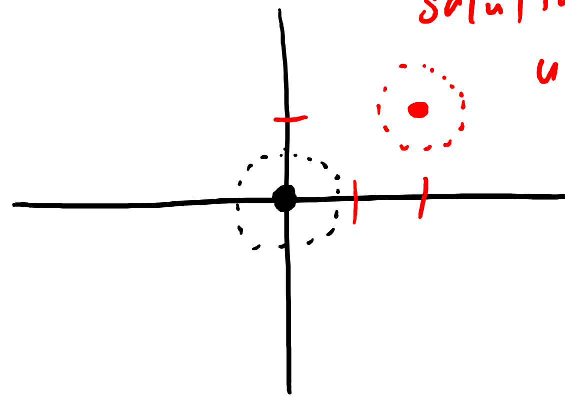


Ex. Show that $\frac{dy}{dx} = xy^{1/2}$, $y(0) = 0$ fails the hypotheses for this theorem.

$$f(x,y) = x y^{1/2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2} x y^{-1/2} = \frac{x}{2\sqrt{y}}$$

$y(2)=1$
solution is
unique

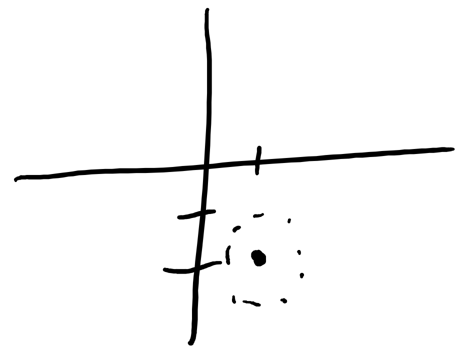


both are not cont. in this region, so a solution does not have to be unique

Ex. Show that $y' = y$, $y(1) = -2$ satisfies the theorem, so the solution we found is the only solution.

$f(x,y) = y$

$\frac{\partial f}{\partial y} = 1$



Note that if we use the IVP $\frac{dy}{dx} = xy^{1/2}, y(2) = 1$ the solution $y = \frac{1}{16}x^4$ is unique.

→ Failing the hypotheses of the theorem doesn't mean that the solution isn't unique, we just can't say for sure that there isn't some other solution.

Mathematical Models (a.k.a. Applications)

A population can be described by the relationship $\frac{dP}{dt} \propto P$

→Means proportional: $\frac{dP}{dt} = kP$

→ Only works for small populations over a short time span, like bacteria

Radioactive decay has the same relationship $\frac{dA}{dt} \propto A$, but here the constant is negative

Ex. Suppose the members of a population are born at a rate proportional to the size of the population (P), and die at a constant rate R. Determine the DE for the population $P(t)$.

$$\frac{dP}{dt} = \underbrace{kP}_{\text{Birth}} - \underbrace{R}_{\text{Death}}$$

IVP: $P(0) = 1{,}000$

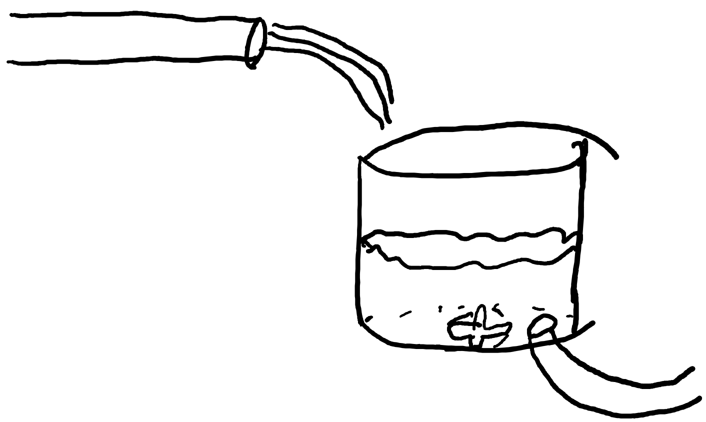
Ex. The acceleration of a jetski is found to be proportional to the sum of its velocity and its distance, $s(t)$, from the beach. Determine a DE for $s(t)$.

$$\frac{d^2s}{dt^2} = k\left(\frac{ds}{dt} + s\right)$$

Mixing salt solutions (brine) with different concentrations can also be represented by a DE.

Let $A(t)$ represent the amount of salt in the tank at time t, then

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt})$$



Ex. A tank initially holds 300 gallons of brine, made up of 50 pounds of salt. A solution with a concentration of 2 pounds of salt per gallon is pumped in at a rate of 3 gallons per minute. While being stirred, fluid is being pumped out at ~~the same rate~~ 5 ~~2~~ gal/min. Express this as an IVP.

$$\frac{dA}{dt} = \underbrace{\frac{3\ \cancel{\text{gal}}}{\text{min}} \cdot \frac{2\ \text{lb.}}{\cancel{\text{gal}}}}_{\text{rate in}} - \underbrace{\frac{5\ \cancel{2}\ \cancel{3}\ \cancel{\text{gal}}}{\text{min}} \cdot \frac{A\ \text{lb.}}{\cancel{300}\ \cancel{\text{gal.}}}}_{\text{rate out}}$$

$\cancel{300+t}$

$300-2t$

$\frac{\text{lbs.}}{\text{min.}}$

$$\boxed{\frac{dA}{dt} = 6 - \frac{A}{100}}$$

$$\boxed{A(0) = 50}$$

Other models we may use

Newton's Law of Cooling/Warming:

$\frac{dT}{dt} \propto T - T_m$ where T_m = surrounding temp.

Newton's Second Law of Motion:

$\frac{d^2 s}{dt^2} = -g, \; s'(0) = v_0, \; s(0) = s_0$

Kirchhoff's Second Law (circuits):

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

$$i(t)=\text{current}$$

$$q(t)=\text{charge}$$

$$E(t)=\text{voltage}$$

$$L=\text{inductance}$$

$$R=\text{resistance}$$

$$C=\text{capacitance}$$

Practice Problems

Section 1.3

Problems 10, 16, 19, 25