Integration Review

Ex. $\int x^3 \cos\left(x^4+2\right)dx$

$= \int \cos u \, \frac{1}{4} du$

$= \frac{1}{4} \sin u + C$

$= \frac{1}{4} \sin\left(x^4+2\right) + C$

$$\boxed{\begin{aligned} u &= x^4+2 \\ du &= 4x^3 dx \\ \tfrac{1}{4} du &= x^3 dx \end{aligned}}$$

Ex. $\int x \sin x dx$

$uv - \int v du$

$$\boxed{\begin{array}{ll} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{array}}$$

$= -x\cos x - \int -\cos x \, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C$

Ex. $\int \cos^3 x\,dx$

$$= \int \cos^2 x \; \underbrace{\cos x\, dx}$$

$$= \int (1 - \sin^2 x) \cos x\, dx$$

$$= \int (1 - u^2)\, du$$

$$= u - \frac{1}{3}u^3 + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x\, dx \end{aligned}}$$

$$\int \cos^2 x\, dx$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Ex. $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$$\boxed{\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\, d\theta \end{aligned}}$$

$$9-x^2$$
$$9-9\sin^2\theta$$
$$9(1-\sin^2\theta)$$
$$9\cos^2\theta$$

$$= \int \frac{\sqrt{\cancel{9}\cos^2\theta}}{\cancel{9}\sin^2\theta}\,\cancel{3}\cos\theta\, d\theta$$

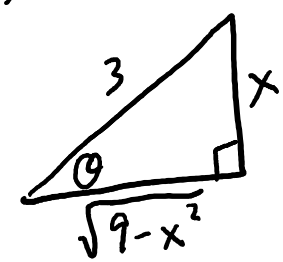
$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$\sin\theta = \frac{x}{3}$$



$$\theta = \sin^{-1}\left(\frac{x}{3}\right)$$

$$1-\sin^2\theta = \cos^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\sec^2\theta - 1 = \tan^2\theta$$

Ex. $\displaystyle\int \frac{9x-2}{2x^2-x}\,dx = \int \frac{2}{x} + \frac{5}{2x-1}\,dx$

$x(2x-1)$

$= 2\ln|x| + \frac{5}{2}\ln|2x-1| + C$

$$\frac{9x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$$9x-2 = A(2x-1) + Bx$$

$x=0$: $-2 = A(-1) \longrightarrow A = 2$

$x=\frac{1}{2}$: $\frac{9}{2} - 2 = B\left(\frac{1}{2}\right)$

$\frac{5}{2} = B\left(\frac{1}{2}\right) \rightarrow B = 5$

Ex. $\displaystyle\int \frac{x^3+x}{x-1}\,dx = \int x^2 + x + 2 + \frac{2}{x-1}\,dx$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C$$

$$
\begin{array}{r|l}
 & x^2 + x + 2 + \dfrac{2}{x-1} \\
\hline
x-1 & x^3 \qquad\quad + x \\
 & -(x^3 - x^2) \\
\cline{2-2}
 & \quad x^2 + x \\
 & \quad -(x^2 - x) \\
\cline{2-2}
 & \qquad\quad 2x \\
 & \qquad -(2x - 2) \\
\cline{2-2}
 & \qquad\qquad 2
\end{array}
$$

Separable Variables

Ex. Solve $\frac{dy}{dx} = x^2 - 3$

$$y = \frac{1}{3}x^3 - 3x + C$$

A first-order DE is called <u>separable</u> if it can be written $\frac{dy}{dx} = f(x)g(y)$

→To solve these, we'll isolate the variables.

→Treat $\frac{dy}{dx}$ like a fraction.

Ex. Solve $\dfrac{dy}{dx} = \dfrac{y}{1+x}$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$e^{\ln|y|} = e^{(\ln|1+x| + C)} \quad \leftarrow \text{ implicit}$$

$$|y| = e^{\ln|1+x|} \cdot e^{C}$$

$$y = D|1+x| \quad \leftarrow \text{ explicit}$$

$$D = \pm e^{C}$$

<u>Ex.</u> Solve $\dfrac{dy}{dx} = 3y$

$$\int \frac{1}{y}\,dy = \int 3\,dx$$

$$\ln|y| = 3x + C$$

$$e \qquad e$$

$$e^{3x+C}$$

$$y = De^{3x}$$

Ex. Solve the IVP $\dfrac{dy}{dx}=\dfrac{-x}{y},\ y(4)=-3$ $(4,-3)$

$$\int y\,dy=\int -x\,dx$$

$$\frac{1}{2}y^2=-\frac{1}{2}x^2+C$$

$$y^2=-x^2+D$$

$$y=\pm\sqrt{D-x^2}$$

$$-3=\pm\sqrt{D-4^2}$$

$$9=D-16$$

$$D=25$$

$$\boxed{y=-\sqrt{25-x^2}}$$

$$-5<x<5$$

$\underline{\text{Ex.}}$ Solve the IVP

$$\left(e^{2y}-y\right)\cos x\frac{dy}{dx}=e^{y}\sin 2x,\ y(0)=0$$

$$\frac{e^{2y}-y}{e^{y}}\,dy = \frac{\overset{2\sin x\cos x}{\cancel{\sin 2x}}}{\cos x}\,dx$$

$$\int\left(e^{y}-ye^{-y}\right)dy=\int 2\sin x\,dx$$

$$e^{y}-\left(-ye^{-y}-e^{-y}\right)=-2\cos x+C$$

$$1+0+1=-2\cdot 1+C$$

$$4=C$$

$$e^{y}+ye^{-y}+e^{-y}=-2\cos x+4$$

$$\int ye^{-y}\,dy$$

$$\boxed{\begin{array}{ll} u=y & dv=e^{-y} \\ du=dy & v=-e^{-y}\end{array}}$$

$$=-ye^{-y}-\int -e^{-y}\,dy$$

$$=-ye^{-y}+\int e^{-y}\,dy$$

$$=-ye^{-y}-e^{-y}$$

If we are asked to find an antiderivative that we can’t do by hand, we need to express the answer as an integral function.

Ex. Solve the IVP $\dfrac{dy}{dx} = e^{-x^2}$, $y(3) = 5$

$$y = \int_3^x e^{-t^2}\,dt + C$$

$$5 = \int_3^3 e^{-t^2}\,dt + C$$

$$5 = C$$

$$y = \int_3^x e^{-t^2}\,dt + 5$$

This is called a nonelementary function.

When we divide both sides of the equation, we may lose a solution where the divisor is zero.

<u>Ex.</u> Solve $\dfrac{dy}{dx} = y^2 - 4$

$$\int \frac{1}{y^2-4}\,dy = \int dx$$

$$\int \frac{1/4}{y-2} - \frac{1/4}{y+2}\,dy = \int dx$$

$$\frac{1}{4}\Big[\ln|y-2| - \ln|y+2|\Big] = x + C$$

$$\cancel{\ln}\left|\frac{y-2}{y+2}\right| = 4x + D$$

$$e \qquad e$$

$$1 - \frac{4}{y+2} = Fe^{4x}$$

$$\frac{-4}{y+2} = Fe^{4x} - 1$$

$$\frac{y+2}{-4} = \frac{1}{Fe^{4x}-1}$$

$$y + 2 = \frac{-4}{Fe^{4x}-1}$$

$$y = \frac{-4}{Fe^{4x}-1} - 2$$

$$\frac{1}{(y+2)(y-2)} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$1 = A(y+2) + B(y-2)$$

$y=2$: $1 = 4A \rightarrow A = \frac{1}{4}$

$y=-2$: $1 = -4B \rightarrow B = -\frac{1}{4}$

$$\begin{array}{r} 1 - \frac{4}{y+2} \\ y+2\,\overline{)\,y-2\quad} \\ -(y+2) \\ \hline -4 \end{array}$$

$$y = \frac{-4}{Fe^{4x} - 1} - 2$$

$y(0) = -2$

$$-2 = \frac{-4}{F - 1} - 2$$

$$0 = \frac{-4}{F-1}$$

$0 = -4$???

$y^2 - 4 = 0$

$$y = 2 \xrightarrow[\text{particular}]{F = 0} y = \frac{-4}{0 - 1} - 2 = 2$$

$$y = -2 \xrightarrow[\text{singular}]{F = ??}$$

$$y = \begin{cases} \dfrac{-4}{Fe^{4x} - 1} - 2 & \text{if init. value has } y \neq -2 \\ -2 & \text{if init. value has } y = -2 \end{cases}$$

Did we lose any solutions? Are they particular or singular?